

ON SCHUR SUPERFUNCTORS

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ABSTRACT. We introduce super-analogues of the Schur functors defined by Akin, Buchsbaum and Weyman. These *Schur superfunctors* may be viewed as characteristic-free analogues of the finite dimensional atypical irreducible modules over the Lie superalgebra $\mathfrak{gl}(m|n)$ studied by Berele and Regev. Our construction realizes Schur superfunctors as objects of a certain category of strict polynomial superfunctors. We show that Schur superfunctors are indecomposable objects of this category. Another aim is to provide a decomposition of *Schur bisuperfunctors* in terms of tensor products of Schur superfunctors.

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INTRODUCTION

Let \mathbb{k} be an arbitrary field. Then it is well known that irreducible polynomial representations of the general linear group scheme, GL_n , correspond to partitions, λ , of length smaller than or equal to n . Furthermore, each such irreducible may be obtained as the simple head (or cosocle) of a corresponding *Schur* (or *costandard*) *module*, $S_\lambda(\mathbb{k}^n)$, which is essentially an induced module from a Borel subgroup of GL_n , as defined in [Ja]. (See also [G] for a definition in terms of Schur algebras.) In [ABW], Akin, Buchsbaum and Weyman gave an explicit construction of these Schur modules which surprisingly uses only rudimentary multilinear algebra stated in terms of familiar Hopf algebras. The goal of this paper is to define and study super-analogues of this construction.

More specifically, let V be a finite dimensional \mathbb{k} -vector space, and consider the Hopf algebra DV (resp. ΛV , $\text{Sym} V$) of divided (resp. exterior, symmetric) powers of V . Also, given a partition $\lambda = (\lambda_1, \dots, \lambda_t)$, let λ' denote its transpose. Then in [ABW], the Schur module $S_\lambda(V)$ is defined to be the image of a certain linear map

$$\theta_\lambda(V) : \Lambda^{\lambda'} V \rightarrow V^{\otimes d} \rightarrow \text{Sym}^\lambda V,$$

where $\Lambda^{\lambda'}(V)$, $\text{Sym}^\lambda(V)$ are tensor products of exterior and symmetric powers, respectively. Dually, the Weyl (or co-Schur) module $W_\lambda(V)$ is defined as the image of a composition

$$\theta'_\lambda(V) : D^{\lambda'} V \rightarrow V^{\otimes d} \rightarrow \Lambda^\lambda V.$$

In both cases, the linear maps are obtained by composing a permutation of the multiplication with iterations of the comultiplication in the respective Hopf algebras.

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Now let $\mathcal{P}_d = \mathcal{P}_{d,\mathbb{k}}$ denote the category of (homogeneous) strict polynomial functors defined by Friedlander and Suslin in [FS]. Let vec denote the category of finite dimensional \mathbb{k} -vector spaces, and let \mathbf{sch} denote the category of all schemes over \mathbb{k} . Then objects of \mathcal{P}_d may be considered as (homogeneous) \mathbf{sch} -enriched endofunctors of vec . In particular, the action on hom-spaces may be described in terms of polynomial equations which are homogeneous of degree d . We recall in Section 2.1 the equivalent but somewhat simpler “linearized” definition of the category \mathcal{P}_d . Typical examples of strict polynomial functors are D^d , Λ^d , and Sym^d which send a vector space to its d -th divided, exterior, or symmetric power, respectively. Since there is a tensor product bifunctor, $- \otimes - : \mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \rightarrow \mathcal{P}_{d_1+d_2}$, we may also consider the objects $D^\lambda, \Lambda^\lambda, \text{Sym}^\lambda \in \mathcal{P}_d$, where λ is a partition such that $|\lambda| := \sum \lambda_i = d$. Evaluation of $T \in \mathcal{P}_d$ at $V \in \text{vec}$ gives rise to a polynomial $GL(V)$ -module, $T(V)$. Furthermore, if $n \geq d$ then \mathcal{P}_d is equivalent to the category of all finite dimensional polynomial GL_n -modules which are homogeneous of degree d .

The naturality of the construction in [ABW] allows a straightforward generalization to the functorial setting. I.e., the linear maps $\theta_\lambda(V)$ and $\theta'_\lambda(V)$ yield natural transformations which are composites of morphisms

$$\Lambda^{\lambda'} \hookrightarrow I^{\otimes d} \twoheadrightarrow \text{Sym}^\lambda \quad \text{and} \quad D^{\lambda'} \hookrightarrow I^{\otimes d} \twoheadrightarrow \Lambda^\lambda$$

in the category \mathcal{P}_d , respectively, where $I^{\otimes d}$ is the d -th tensor product functor. Since \mathcal{P}_d is abelian, one may define the *Schur functor*, S_λ , to be the image of θ_λ , and the *Weyl functor*, W_λ , is the image of θ'_λ in \mathcal{P}_d .

In [Ax], the author defined categories $\text{Pol}_d^{\text{I}}, \text{Pol}_d^{\text{II}}$ whose objects are *polynomial superfunctors* of types I and II. If $m, n \geq d$, then $\text{Pol}_d^{\text{I}}, \text{Pol}_d^{\text{II}}$ are equivalent to the category of finite dimensional (degree d homogeneous) polynomial representations over the algebraic supergroups $GL(m|n)$, $Q(n)$, respectively (see [BrKu], [BrK2] for the definitions of these supergroups). In this paper, we will mostly be concerned with the category Pol_d^{I} of type I polynomial superfunctors. Some results, however, can be transferred to the category Pol_d^{II} via a certain restriction functor, $\text{Pol}_d^{\text{I}} \rightarrow \text{Pol}_d^{\text{II}}$, described in Section 3.1 below.

Typical examples of objects in Pol_d^{I} are given by $I^{\otimes d}, \Gamma^d, S^d$ which are super analogues of the tensor, divided, and symmetric powers considered above. Since tensor products of polynomial superfunctors are well-defined, we may also consider $\Gamma^\lambda, S^\lambda \in \text{Pol}_d^{\text{I}}$, for any partition λ such that $|\lambda| = d$. We will also need to consider the (right) parity change functor $\Pi \in \text{Pol}_1^{\text{I}}$. Given any $T \in \text{Pol}_d^{\text{I}}$, we may then consider both left and right compositions, $\Pi \circ T$ and $T \circ \Pi$, as objects in Pol_d^{I} .

Now suppose λ is a partition with $|\lambda| = d$. We define the Schur superfunctor \hat{S}_λ in Section 3.2 as the object of Pol_d^{I} which is the image of a certain natural transformation $\hat{\theta}_\lambda$. This morphism $\hat{\theta}_\lambda$ is given as a composite of natural transformations of the form

$$\Pi \circ \Gamma^{\lambda'} \circ \Pi \hookrightarrow I^{\otimes d} \twoheadrightarrow S^\lambda \quad (\Gamma^{\lambda'} \circ \Pi \hookrightarrow I^{\otimes d} \twoheadrightarrow S^\lambda),$$

if d is odd (even). Similarly, we define the Weyl superfunctor \hat{W}_λ to be the image of a natural transformation $\hat{\theta}_\lambda$ given by a composite of morphisms of the form

$$\Gamma^{\lambda'} \hookrightarrow I^{\otimes d} \twoheadrightarrow \Pi \circ S^\lambda \circ \Pi \quad (\Gamma^{\lambda'} \hookrightarrow I^{\otimes d} \twoheadrightarrow S^\lambda \circ \Pi),$$

if d is odd (even). More generally, we will define skew Schur and Weyl superfunctors, $\hat{S}_{\lambda/\mu}$ and $\hat{W}_{\lambda/\mu}$, associated to any skew partition, λ/μ .

In [ABW], the authors also studied *Schur complexes*, which they used to obtain resolutions of certain polynomial GL_n -modules. We consider these Schur complexes as *strict polynomial bifunctors* $SC_{\lambda/\mu}(\ , \)$ associated to skew partitions λ/μ (cf. Definition 2.3.1). Evaluating $SC_{\lambda/\mu}$ at a pair of vector spaces V, W gives (the underlying vector space of) a corresponding Schur complex. We will show in Proposition 3.2.4 that $SC_{\lambda/\mu}(V, W)$ is also isomorphic to (the underlying ordinary vector space of) the evaluation of the Schur superfunctor $\hat{S}_{\lambda/\mu}$ at a certain vector superspace.

In this paper, we also give analogues of several results obtained in [ABW] for Schur functors and Schur complexes. For example, we provide a Standard Basis Theorem (Theorem 3.3.10) for Schur superfunctors using a certain “Straightening Law” (Lemma 3.3.7). We also consider categories \mathbf{biPol}_d^I , \mathbf{biPol}_d^{II} of *strict polynomial bisuperfunctors* in Section 4. After defining the *Schur bisuperfunctors*, $\hat{S}_{\lambda/\mu}^{\mathbf{bi}} \in \mathbf{biPol}_d^I$, for any skew partition λ/μ such that $|\lambda| - |\mu| = d$, we prove a filtration of these superfunctors in Theorem 4.2.5. This result is a super-analogue of Theorem II.4.11 in [ABW], which was recently used to obtain a categorification of Fock space representations in [HTY].

We conclude the paper by showing that the Schur superfunctors \hat{S}_λ are indecomposable objects in the category \mathbf{Pol}_d^I . In order to prove this, we first show in Theorem 5.2.6 that the Schur supermodule $\hat{S}_\lambda(M)$, given by evaluation at a superspace M , is an indecomposable module for the algebraic supergroup $GL(M)$ (or rather of the corresponding Schur superalgebra). On the other hand, the Schur superfunctor, \hat{S}_λ^{II} , is usually not an indecomposable object of the category \mathbf{Pol}_d^{II} . It is thus an interesting problem to define an analogue of Schur superfunctors of type II whose corresponding Schur supermodules are indecomposable $Q(n)$ -supermodules. Such a construction seems to be lacking even in characteristic zero.

In [LZ], La Scala and Zubkov also constructed *costandard modules*, $\nabla(\lambda)$, for the supergroup $GL(M)$. Provided that the size of λ is small enough, these supermodules appear to be isomorphic to our Schur supermodules, $\hat{S}_\lambda(M)$. In particular, the standard basis of $\nabla(\lambda)$ given in [LZ, Theorem 6.6] is in bijection with the standard basis given in Theorem 3.3.10 below.

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1. SUPERALGEBRA PRELIMINARIES

We will assume throughout that \mathbb{k} is an arbitrary field of characteristic not equal 2. Given V, W a pair of \mathbb{k} -vector spaces, the set of all linear maps from V to W is denoted $\mathrm{Hom}(V, W)$.

1.1. Categories enriched in \mathbb{k} -superspaces. A *vector superspace* is a \mathbb{k} -vector space M with \mathbb{Z}_2 -grading, $M = M_0 \oplus M_1$. If $\dim(M_0) = m$ and $\dim(M_1) = n$, we write $\mathrm{sdim}(M) = m|n$. An element of M_0 (resp. M_1) is called *even* (resp. *odd*). We write \underline{M} to denote the underlying ordinary vector space without a \mathbb{Z}_2 -grading. A *subsuperspace* of M is a subspace N of M such that $N = (N \cap M_0) \oplus (N \cap M_1)$. If M and N are superspaces, we consider $\mathrm{Hom}(M, N)$ as a superspace by setting $\mathrm{Hom}(M, N)_\varepsilon$ equal to the set of linear maps $f : M \rightarrow N$ which are \mathbb{Z}_2 -homogeneous of degree ε , i.e. $f(M_\delta) \subset N_{\varepsilon+\delta}$ ($\varepsilon, \delta \in \mathbb{Z}_2$). The \mathbb{k} -linear dual $M^* = \mathrm{Hom}(M, \mathbb{k})$ is a superspace by viewing \mathbb{k} as vector superspace concentrated in degree 0.

The tensor product $M \otimes N$ of superspaces is also a superspace with $(M \otimes N)_0 = M_0 \otimes N_0 \oplus M_1 \otimes N_1$ and $(M \otimes N)_1 = M_0 \otimes N_1 \oplus M_1 \otimes N_0$. There is an even linear isomorphism $\tau_{M,N} : M \otimes N \rightarrow N \otimes M$ of superspaces, given by

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \quad (v \in M, w \in N),$$

called the *supertwist map*.

A *superalgebra* is an associative algebra A which is also a superspace such that $A_\varepsilon A_\delta \subset A_{\varepsilon+\delta}$. As above, \underline{A} denotes the ordinary associative algebra obtained by forgetting the \mathbb{Z}_2 -grading. Given two superalgebras A and B , a *superalgebra homomorphism* $\varphi : A \rightarrow B$ is an even linear map which is an algebra homomorphism; its kernel is a *superideal*, i.e., an ordinary two-sided ideal which is also a subsuperspace. We view the tensor product of superspaces $A \otimes B$ as a superalgebra with multiplication defined by the usual *rule of signs* convention:

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} (aa') \otimes (bb') \quad (a, a' \in A, b, b' \in B). \quad (1)$$

The supertwist map gives an isomorphism $\tau_{A,B} : A \otimes B \cong B \otimes A$ of superalgebras.

Example 1.1.1. The *Clifford superalgebra*, \mathcal{C}_n , is the superalgebra with odd generators c_1, \dots, c_n satisfying: $c_i^2 = 1$ and $c_i c_j = -c_j c_i$, for $1 \leq i \neq j \leq n$. For any $n \geq 1$, there is an isomorphism $\mathcal{C}_n \cong \mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_1$ (n copies).

Let A be a superalgebra. A *right A -supermodule* is a superspace V which is a right A -module in the usual sense, such that $A_\varepsilon V_\delta \subset V_{\varepsilon+\delta}$ for $\varepsilon, \delta \in \mathbb{Z}_2$. One may similarly define *left A -supermodules*. A *homomorphism* $\varphi : V \rightarrow W$ of right (resp. left) A -supermodules V and W is a linear map φ such that

$$\varphi(va) = \varphi(v)a \quad (\text{resp. } \varphi(av) = (-1)^{|\varphi||a|}a\varphi(v)), \quad \forall a \in A, v \in V.$$

Let $\text{Smod } A$ (resp. $A\text{Smod}$) denote the category of all right (resp. left) A -supermodules and A -supermodule homomorphisms. Let $\text{smod } A$ (resp. Asmod) denote the full subcategory of all right (resp. left) A -supermodules which are finite dimensional (over \mathbb{k}). Similarly, write Svec (resp. svec) for the category of all (resp. finite dimensional) \mathbb{k} -vector superspaces. If $V, W \in \text{Asmod}$ (resp. $\text{smod } A$), we let $\text{Hom}_A(V, W)$ denote the set of A -homomorphisms from V to W . Also let $\text{End}_A(V)$ denote the superalgebra of all A -supermodule endomorphisms of V .

Given any superspace M we denote by ΠM the same vector space, but with opposite \mathbb{Z}_2 -grading. For example, we write $\mathbb{k}^{m|n} = \mathbb{k}^m \oplus (\Pi \mathbb{k})^n$. Given a superalgebra A , we define the (*right*) *parity change* functor $\Pi : \text{Smod } A \rightarrow \text{Smod } A$ which sends $V \mapsto \Pi V$, such that ΠV has the same right action: $v \cdot a = va$ ($a \in A, v \in V$). On a right A -supermodule homomorphism $\varphi : V \rightarrow W$, we set $\Pi(\varphi) = (-1)^{|\varphi|}\varphi$ as a linear map.

We say a category \mathcal{V} is an *Svec-enriched category* if the hom-sets

$$\text{hom}_{\mathcal{V}}(V, W) \quad (V, W \in \mathcal{V})$$

are \mathbb{k} -superspaces, while composition is bilinear and *even*. I.e., if $U, V, W \in \mathcal{V}$, then composition induces an even linear map:

$$\text{hom}_{\mathcal{V}}(V, W) \otimes \text{hom}_{\mathcal{V}}(U, V) \rightarrow \text{hom}_{\mathcal{V}}(U, W).$$

As usual if V, W are isomorphic in \mathcal{V} , we write $V \cong W$. If there is an even isomorphism $\varphi : V \cong W$ (i.e., $\varphi \in \text{hom}_{\mathcal{V}}(V, W)_0$), then we use the notation

$$V \simeq W.$$

Let \mathcal{V}_{ev} denote the subcategory of \mathcal{V} consisting of the same objects but only even homomorphisms.

For a superalgebra A , the category $A\text{Smod}$ ($\text{Smod } A$) is naturally an Svec-enriched category, as is the full subcategory Asmod ($\text{smod } A$). Further, the even subcategories $(A\text{Smod})_{\text{ev}}$, etc. are abelian categories in the usual sense. All functors between the Svec-enriched categories which we consider will send even morphisms to even morphisms. They will thus give rise to corresponding functors between the underlying even subcategories.

Definition 1.1.2. Suppose $\mathcal{V}, \mathcal{V}'$ are Svec-enriched categories. An *even* functor $F : \mathcal{V} \rightarrow \mathcal{V}'$ is called *\mathbb{k} -linear* (or *Svec-enriched*) if the function

$$F_{V,W} : \text{hom}_{\mathcal{V}}(V, W) \rightarrow \text{hom}_{\mathcal{V}'}(FV, FW)$$

is a linear map for any pair of objects $V, W \in \mathcal{V}$. Suppose that $F, G : \mathcal{V} \rightarrow \mathcal{V}'$ are a pair of *\mathbb{k} -linear functors*. Then an *even* (resp. *odd*) *Svec-enriched natural transformation* $\eta : F \rightarrow G$ consists of a collection of even (resp. odd) linear maps

$$\eta(V) \in \text{hom}_{\mathcal{V}'}(FV, GV) \quad (\forall V \in \mathcal{V})$$

such that for a given $\varphi \in \text{hom}_{\mathcal{V}}(V, W)$ we have

$$G(\varphi) \circ \eta(W) = (-1)^{|\varphi||\eta(V)|} \eta(W) \circ F(\varphi).$$

In general, an *Svec-enriched natural transformation*, $\eta : F \rightarrow G$, is defined to be a collection of linear maps, $\eta(V) = \eta_0(V) \oplus \eta_1(V)$, for all $V \in \mathcal{V}$, such that

$$\eta_\varepsilon(V) \in \text{hom}_{\mathcal{V}'}(FV, GV)_\varepsilon \quad (\varepsilon \in \mathbb{Z}_2)$$

and η_0 (resp. η_1): $F \rightarrow G$ is an even (resp. odd) Svec-enriched natural transformation.

Definition 1.1.3 (Cartesian product of Svec-enriched categories). Given two Svec-enriched categories \mathcal{V}, \mathcal{W} we may form the direct product $\mathcal{V} \times \mathcal{W}$, which is the Svec-enriched category with pairs (V, W) ($V \in \mathcal{V}, W \in \mathcal{W}$) as objects and morphisms

$$\text{hom}_{\mathcal{V} \times \mathcal{W}}((V, W), (V', W')) = \text{hom}_{\mathcal{V}}(V, V') \otimes \text{hom}_{\mathcal{W}}(W, W').$$

The composition of morphisms follows the rule of signs convention (1) stated above.

In the above definition, notice that we have chosen the hom-spaces to be tensor products instead of Cartesian products. In this way, we may consider only even, \mathbb{k} -linear functors: $\mathcal{V} \times \mathcal{W} \rightarrow \mathcal{U}$, instead of bilinear functors.

1.2. Hopf superalgebras. A *cosuperalgebra* is a superspace C with the structure of \mathbb{k} -coalgebra such that the comultiplication $\Delta : C \otimes C \rightarrow C$ and the counit $\epsilon : C \rightarrow \mathbb{k}$ are even linear maps. If C and D are cosuperalgebras, the tensor product $C \otimes D$ is a cosuperalgebra with comultiplication

$$\Delta_{C \otimes D} = 1 \otimes \tau_{C,D} \otimes 1 \circ (\Delta_C \otimes \Delta_D).$$

A homomorphism $\varphi : C \rightarrow D$ of cosuperalgebras is an even linear map which is a homomorphism of coalgebras in the usual sense.

By a *bisuperalgebra*, we mean a superalgebra which is also a cosuperalgebra such that the comultiplication and counit are superalgebra homomorphisms. E.g., if A is a bisuperalgebra, then the following diagrams are commutative:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\Delta \otimes \Delta} & A \otimes A \otimes A \otimes A \\ m_A \downarrow & & \downarrow m_{A \otimes A} \\ A & \xrightarrow{\Delta} & A \otimes A, \end{array} \quad \begin{array}{ccc} A \otimes A & \xrightarrow{m_A} & A \\ \epsilon \otimes \epsilon \searrow & & \downarrow \epsilon \\ & & \mathbb{k}. \end{array} \quad (2)$$

(Recall that $m_{A \otimes A}$ is defined to be $m_A \otimes m_A \circ (1 \otimes \tau_{A,A} \otimes 1)$.) A morphism $\varphi : A \rightarrow B$ of bisuperalgebras is a superalgebra map which is also a map of cosuperalgebras.

A *Hopf superalgebra* is a bisuperalgebra A which is also a Hopf algebra, such that the *antipode*, $\iota : A \rightarrow A$, is an even linear map.

Definition 1.2.1. We defined a $\mathbb{Z}_{\geq 0}$ -graded superspace to be a superspace $M = \bigoplus_{d=0}^{\infty} M^d$ such that each M^d is a finite dimensional subsuperspace. A $\mathbb{Z}_{\geq 0}$ -graded superspace M is *connected* if $M^0 = \mathbb{k}$.

Note that the tensor product of $\mathbb{Z}_{\geq 0}$ -graded superspaces, $M \otimes N$, is naturally $\mathbb{Z}_{\geq 0}$ -graded, with $(M \otimes N)^d = \bigoplus_{i+j=d} M^i \otimes N^j$.

Definition 1.2.2. Suppose A is a (co)superalgebra whose underlying superspace is $\mathbb{Z}_{\geq 0}$ -graded. Then A is a *graded (co)superalgebra* if the multiplication $m : A \otimes A \rightarrow A$ (resp. comultiplication $\Delta : A \rightarrow A \otimes A$) is a $\mathbb{Z}_{\geq 0}$ -homogeneous map. A *graded bisuperalgebra* is a bisuperalgebra which is graded both as a superalgebra and as a cosuperalgebra simultaneously.

Define the *graded dual*, $A^{\text{gr},*}$, of a graded bisuperalgebra, $A = \bigoplus_{d=0}^{\infty} A^d$, to be the bisuperalgebra

$$A^{\text{gr},*} = \bigoplus_{d=0}^{\infty} (A^d)^*,$$

with multiplication m^{gr} and comultiplication Δ^{gr} defined to be the maps which are dual to comultiplication Δ and multiplication m , respectively.

Remark 1.2.3. By Takeuchi's antipode formula (cf. [Ta, Lem. 14] or, e.g., [HGK, Prop. 3.8.8]), any connected ($\mathbb{Z}_{\geq 0}$)-graded bialgebra is automatically a Hopf algebra. Using this formula, it may be checked that if A is a connected bisuperalgebra, then both A and $A^{\text{gr},*}$ are also Hopf superalgebras.

1.3. The symmetric superalgebra. Let M be a vector superspace. For $d \in \mathbb{Z}_{\geq 1}$, we write $M^{\otimes d} = M \otimes \cdots \otimes M$ (d copies) and $M^{\otimes 0} = \mathbb{k}$. The *tensor superalgebra*, $\mathsf{T}M$, is the tensor algebra

$$\mathsf{T}M = \bigoplus_{d \geq 0} M^{\otimes d}$$

regarded as a vector superspace. It is the free associative superalgebra generated by M . Hence, there is a unique superalgebra map $\Delta : \mathsf{T}M \rightarrow \mathsf{T}M \otimes \mathsf{T}M$ such that: $v \mapsto v \otimes 1 + 1 \otimes v$. Also, let $\epsilon : \mathsf{T}M \rightarrow \mathbb{k}$ be such that $\epsilon(1) = 1$ and $\epsilon(x) = 0$, if $x \in M^{\otimes d}$ with $d > 0$. Then these maps makes $\mathsf{T}M$ into a bisuperalgebra. The fact that $\mathsf{T}M$ is a Hopf superalgebra follows by Remark 1.2.3.

A superalgebra A is called *commutative* if $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$. The *symmetric superalgebra*, $\mathsf{S}M$, is the free commutative superalgebra generated by M . Explicitly, $\mathsf{S}M = \bigoplus_{d \geq 0} S^d M$ is the quotient of $\mathsf{T}M$ by the super ideal

$$\mathcal{I} = \langle x \otimes y - (-1)^{|x||y|} y \otimes x; \ x, y \in M \rangle.$$

Since the tensor product of commutative superalgebras is also commutative, there is a unique homomorphism $\Delta : \mathsf{S}M \rightarrow \mathsf{S}M \otimes \mathsf{S}M$ of superalgebras such that: $v \mapsto v \otimes 1 + 1 \otimes v$. Define the counit $\epsilon : \mathsf{S}M \rightarrow \mathbb{k}$ in the same way as above. Together these maps give $\mathsf{S}M$ the structure of Hopf superalgebra (again using Remark 1.2.3).

Notice that $\mathsf{S}(\)$ satisfies the *exponential property*, since there is an isomorphism:

$$\mathsf{S}(M) \otimes \mathsf{S}(N) \cong \mathsf{S}(M \oplus N),$$

which is given by: $x \otimes y \mapsto xy$, for any $x \in \mathsf{S}M, y \in \mathsf{S}N$. In particular, we have $\mathsf{S}(M) \cong \mathsf{S}(M_0) \otimes \mathsf{S}(M_1)$. By forgetting the \mathbb{Z}_2 -grading, we then have an isomorphism

$$\underline{\mathsf{S}(M)} \cong \text{Sym } \underline{M_0} \otimes \Lambda \underline{M_1} \quad (3)$$

of ordinary (graded) associative algebras, where $\text{Sym}(\)$ and $\Lambda(\)$ denote the symmetric and exterior algebras of a vector space, respectively.

Suppose now that $M \in \text{vec}$. Let $\mathcal{X} = (X_1, \dots, X_m)$ be a basis of M_0 and $\mathcal{Y} = (Y_1, \dots, Y_n)$ a basis of M_1 . If we write the multiplication in $\mathsf{S}M$ simply by juxtaposition, then it follows from (3) that the set of elements

$$\{X_1^{d_1} \cdots X_m^{d_m} Y_1^{e_1} \cdots Y_n^{e_n}; \ d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}, \ e_1, \dots, e_n \in \{0, 1\}, \text{ and } \sum d_i + \sum e_j = d\} \quad (4)$$

forms a basis of $\mathsf{S}^d M$.

1.4. The divided powers superalgebra. Let $M \in \text{vec}$ and suppose $d \in \mathbb{Z}_{\geq 0}$. Then we define the *d-th divided power* of M to be the superspace: $\Gamma^d M = \mathsf{S}^d(M^*)^*$.

Suppose as above that $\mathcal{X} = (X_1, \dots, X_m)$ and $\mathcal{Y} = (Y_1, \dots, Y_n)$ are bases of M_0 and M_1 , which are ordered by indices. Fix some total order on their union $\mathcal{Z} = \mathcal{X} \sqcup \mathcal{Y}$ which preserves the relative orders in \mathcal{X} and \mathcal{Y} respectively. We write the elements of \mathcal{Z} as Z_1, \dots, Z_{m+n} with the total order again indicated by indices. Now \mathcal{Z} is a \mathbb{Z}_2 -graded set, with $\mathcal{Z}_0 = \mathcal{X}$ and $\mathcal{Z}_1 = \mathcal{Y}$. We write $|Z_i| = \varepsilon$ if $Z_i \in \mathcal{Z}_\varepsilon$, for $i = 1, \dots, m+n$ and $\varepsilon = 0, 1$.

Let $\check{\mathcal{Z}} = (\check{Z}_1, \dots, \check{Z}_{m+n})$ denote the basis of M^* which is dual to \mathcal{Z} . Suppose $\alpha \in (\mathbb{Z}_{\geq 0})^{m+n}$ is such that $|\alpha| = \sum \alpha_i = d$. Then we denote by

$$Z^{(\alpha)} = Z_1^{(\alpha_1)} \cdots Z_{m+n}^{(\alpha_{m+n})}$$

the element of $\Gamma^d M$ which is dual to the monomial $\check{Z}_1^{\alpha_1} \cdots \check{Z}_{m+n}^{\alpha_{m+n}}$ of $\mathsf{S}^d(M^*)$. Since $\mathsf{S}^*(M^*)$ is commutative and (4) is a basis of $\mathsf{S}^d(M^*)$, it follows that

$$\{Z^{(\alpha)} : |\alpha| = d \text{ and } 0 \leq \alpha_i \leq 1 \text{ if } |Z_i| = 1\}$$

gives a basis of $\Gamma^d M$.

Let us consider an alternative way to denote this basis for $\Gamma^d M$. We write $I(m|n, d)$ to denote the set of all functions $\mathbf{i} : \{1, \dots, d\} \rightarrow \{1, \dots, m+n\}$. We may identify any $\mathbf{i} \in I(m|n, d)$ as a

sequence $\mathbf{i} = (i_1, \dots, i_d)$ of elements $i_k = \mathbf{i}(k) \in \{1, \dots, m+n\}$. For each $\mathbf{i} \in I(m|n, d)$ there is a corresponding element $Z^{(\mathbf{i})} \in \Gamma^d M$ which is dual to the element

$$\check{Z}_{i_1} \check{Z}_{i_2} \cdots \check{Z}_{i_d} \in S^d M.$$

If $\mathbf{i} \in I(m|n, d)$, we let $wt(\mathbf{i}) \in (\mathbb{Z}_{\geq 0})^{m+n}$ denote the sequence obtained by setting

$$wt(\mathbf{i})_s = \#\{1 \leq k \leq d : i_k = s\},$$

for $s = 1, \dots, m+n$.

We say that $\mathbf{i} \in I(m|n, d)$ is \mathcal{Z} -restricted if: $wt(\mathbf{i})_s \leq 1$ whenever $|Z_s| = 1$. Note that the symmetric group \mathfrak{S}_d acts on $I(m|n, d)$ by composition, i.e. $(\mathbf{i}.\sigma)(k) = \mathbf{i}(\sigma k)$ for any $\sigma \in \mathfrak{S}_d$. We define the *standardization* of \mathbf{i} to be the sequence $st(\mathbf{i}) = st_{\mathcal{Z}}(\mathbf{i})$ obtained by rearranging the entries of \mathbf{i} into non-decreasing order. I.e., $st(\mathbf{i}) = \mathbf{i}.\sigma$ for some $\sigma \in \mathfrak{S}_d$ such that $i_{\sigma 1} \leq \dots \leq i_{\sigma d}$. We say that \mathbf{i} is *standardized* if $\mathbf{i} = st(\mathbf{i})$. If \mathbf{i} is standardized, then $Z^{(\mathbf{i})} = Z^{(wt(\mathbf{i}))}$, and $Z^{(\mathbf{i})} \neq 0$ if and only if $\mathbf{i} \in I(m|n, d)$ is \mathcal{Z} -restricted. It follows that

$$\{Z^{(\mathbf{i})} : \mathbf{i} \in I(m|n, d) \text{ is } \mathcal{Z}\text{-standardized and } \mathcal{Z}\text{-restricted}\}$$

gives another way to denote our basis for $\Gamma^d M$.

For any $\mathbf{i} = (i_1, \dots, i_d) \in I(m|n, d)$, let us write $|\mathbf{i}|_{\mathcal{Z}}$ to denote the sequence in $(\mathbb{Z}_2)^{m+n}$ given by: $|\mathbf{i}|_{\mathcal{Z}} = (|Z_{i_1}|, \dots, |Z_{i_{m+n}}|)$. Given $\varepsilon \in (\mathbb{Z}_2)^d$ and $\sigma \in \mathfrak{S}_d$, we let

$$\text{sgn}(\varepsilon, \sigma) = \prod_{\substack{1 \leq s < t \leq d \\ \sigma^{-1}s > \sigma^{-1}t}} (-1)^{\varepsilon_s \varepsilon_t}$$

We then set $\text{sgn}_{\mathcal{Z}}(\mathbf{i}; \sigma) = \text{sgn}(|\mathbf{i}|_{\mathcal{Z}}, \sigma)$. If $\mathbf{i} = \mathbf{j}.\sigma$, then we also write $\text{sgn}_{\mathcal{Z}}(\mathbf{i}; \mathbf{j}) = \text{sgn}_{\mathcal{Z}}(\mathbf{i}; \sigma)$.

There is also a unique right action of the symmetric group \mathfrak{S}_d on the tensor power $M^{\otimes d}$ of a superspace such that each transposition $(i \ i+1)$ for $1 \leq i \leq d-1$ acts by:

$$(v_1 \otimes \cdots \otimes v_d).(i \ i+1) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d,$$

for any $v_1, \dots, v_d \in M$ with v_i, v_{i+1} being \mathbb{Z}_2 -homogeneous. We denote by $Z^{\mathbf{i}}$ the element of $M^{\otimes d}$ given by $Z^{\mathbf{i}} = Z_{i_1} \otimes \cdots \otimes Z_{i_d}$. Then $M^{\otimes d}$ has as a basis $\{Z^{\mathbf{i}} | \mathbf{i} \in I(m|n, d)\}$, and the above action may be given explicitly by

$$Z^{\mathbf{i}}.\sigma = \text{sgn}_{\mathcal{Z}}(\mathbf{i}; \sigma) Z^{\mathbf{i}.\sigma}, \quad (5)$$

for any $\sigma \in \mathfrak{S}_d$.

Notice that the symmetric power is the coinvariant superspace, $S^d M = (M^{\otimes d})_{\mathfrak{S}_d}$, with respect to the above action. This means that there is a canonical even isomorphism

$$\text{Hom}_{\mathbb{k}\mathfrak{S}_d}(M^{\otimes d}, N) \simeq \text{Hom}_{\mathbb{k}\mathfrak{S}_d}(S^d M, N) \quad (6)$$

for any $N \in \text{svec}$ considered as a trivial \mathfrak{S}_d -module. Now there is also a right action of \mathfrak{S}_d on the dual space $(M^{\otimes d})^*$ given by $(f.\sigma)(v) = f(v.\sigma^{-1})$, for $f \in (M^{\otimes d})^*, v \in M^{\otimes d}$ and $\sigma \in \mathfrak{S}_d$. Further $(M^*)^{\otimes d} \simeq (M^{\otimes d})^*$ as \mathfrak{S}_d module. It then follows from the definition of $\Gamma^d M$ and (6) that we have a canonical even isomorphism

$$\text{Hom}_{\mathbb{k}\mathfrak{S}_d}(N, M^{\otimes d}) \simeq \text{Hom}_{\mathbb{k}\mathfrak{S}_d}(N, \Gamma^d M)$$

for any $M \in \text{svec}$. Hence the d -th divided power is isomorphic to the subsuperspace of invariants: $\Gamma^d M \simeq (M^{\otimes d})^{\mathfrak{S}_d}$.

We define the *divided powers superalgebra* to be the dual Hopf superalgebra:

$$\Gamma M = (S^* M^*)^{\text{gr},*} = \bigoplus_{d \geq 0} \Gamma^d M.$$

Let us denote the multiplication of elements $f, g \in \Gamma M$ by $f * g$. Then $\Gamma(\)$ also satisfies the exponential property. This follows by duality from the exponential property for $S^*(\)$, since there is an isomorphism

$$\Gamma(M) \otimes \Gamma(N) \cong \Gamma(M \oplus N), \quad (7)$$

given by $x \otimes y \mapsto x * y$, for all $x \in \Gamma(M)$, $y \in \Gamma(N)$. We also have an isomorphism of ordinary (graded) algebras

$$\underline{\Gamma M} \cong D \underline{M_0} \otimes \Lambda \underline{M_1}, \quad (8)$$

where $D(\cdot)$ denotes the usual divided power algebra of a vector space.

Suppose $\mathbf{i} \in I(m|n, d)$ and $\mathbf{j} \in I(m|n, d')$. Then we define $\mathbf{i} \vee \mathbf{j} \in I(m|n, d + d')$ to be the element

$$\mathbf{i} \vee \mathbf{j} = (i_1, \dots, i_d, j_1, \dots, j_{d'}).$$

Let us write $l(\mathbf{i}) = d$ if $\mathbf{i} \in I(m|n, d)$ for some $d > 0$, and we set

$$I(m|n) = \bigcup_{d \geq 1} I(m|n, d).$$

Then $(\mathbf{i}, \mathbf{j}) \mapsto \mathbf{i} \vee \mathbf{j}$ defines a binary operation $I(m|n) \times I(m|n) \rightarrow I(m|n)$.

Lemma 1.4.1. *Let $\mathcal{Z} = \mathcal{Z}_0 \sqcup \mathcal{Z}_1 = (Z_1, \dots, Z_{m+n})$ be a \mathbb{Z}_2 -homogeneous basis of $M = M_0 \oplus M_1$ as above. Suppose $\mathbf{i} \in I(m|n, d)$, $\mathbf{j} \in I(m|n, d')$ are both standardized.*

- (i) *Let $\Delta : \Gamma^{r+s} M \rightarrow \Gamma^r M \otimes \Gamma^s M$ denote the components of the comultiplication $\Delta : \Gamma M \rightarrow \Gamma M \otimes \Gamma M$. These components are given explicitly by:*

$$\Delta(Z^{(\mathbf{i})}) = \sum_{\text{st}(\mathbf{i}^1 \vee \mathbf{i}^2) = \mathbf{i}} \text{sgn}_{\mathcal{Z}}(\mathbf{i}^1 \vee \mathbf{i}^2; \mathbf{i}) Z^{(\mathbf{i}^1)} \otimes Z^{(\mathbf{i}^2)},$$

summing over ordered pairs of standardized sequences $\mathbf{i}^1 \in I(m|n, r)$, $\mathbf{i}^2 \in I(m|n, s)$.

- (ii) *Multiplication on basis elements in ΓM satisfies:*

$$Z^{(\mathbf{i})} * Z^{(\mathbf{j})} = \begin{cases} C Z^{(\mathbf{i} \vee \mathbf{j})} & \text{if } \mathbf{i} \vee \mathbf{j} \text{ is } \mathcal{Z}\text{-strict} \\ 0 & \text{if } \mathbf{i} \vee \mathbf{j} \text{ is not } \mathcal{Z}\text{-strict,} \end{cases}$$

where C is some positive integer such that $C = 1$ if and only if $\text{Im}(\mathbf{i}) \cap \text{Im}(\mathbf{j}) = \emptyset$.

Proof. The lemma follows from (8) and the usual properties of exterior algebras and divided powers of ordinary vectors spaces, as described in [P, Sec. 1], [Wey, Chap. 1]. \square

Note that there is a surjective map, $m : M^{\otimes d} \twoheadrightarrow S^d M$, given by d -fold multiplication in the symmetric superalgebra. By duality and some abuse of notation, we have a corresponding embedding $\Delta : \Gamma^d M \hookrightarrow M^{\otimes d}$ given by applying successive iterations of comultiplication. This embedding is given explicitly as follows.

If $\mathbf{i} \in I(m|n, d)$, then let us denote the stabilizer of \mathbf{i} in \mathfrak{S}_d by $\mathfrak{S}_{\mathbf{i}} = \{\sigma \in \mathfrak{S}_d : \mathbf{i} \cdot \sigma = \mathbf{i}\}$. Then we have:

$$\Delta(Z^{(\mathbf{i})}) = \sum_{\sigma \in \mathfrak{S}_d / \mathfrak{S}_{\mathbf{i}}} Z^{\mathbf{i} \cdot \sigma} = \sum_{\sigma \in \mathfrak{S}_d / \mathfrak{S}_{\mathbf{i}}} \text{sgn}_{\mathcal{Z}}(\mathbf{i}; \sigma) Z^{\mathbf{i} \cdot \sigma}, \quad (9)$$

where the last equality comes from (5). By the injectivity of $\Delta : \Gamma^d M \hookrightarrow M^{\otimes d}$, we then have

$$Z^{(\mathbf{i})} = \text{sgn}_{\mathcal{Z}}(\mathbf{j}; \sigma) Z^{(\mathbf{j})}$$

in $\Gamma^d M$, for any $\mathbf{j} \in I(m|n, d)$ such that there exists some $\sigma \in \mathfrak{S}_d$, with $\mathbf{i} = \mathbf{j} \cdot \sigma$.

Notice also from the commutativity of diagram (2) that we have $\Delta(x * y) = \Delta(x) * \Delta(y)$ for any $x, y \in \Gamma M$. In particular,

$$\Delta(x * y) = \sum_{\sigma \in \mathfrak{S}_{d+e} / \mathfrak{S}_d \times \mathfrak{S}_e} (\Delta(x) \otimes \Delta(y)) \cdot \sigma, \quad (10)$$

if $x \in \Gamma^d M$ and $y \in \Gamma^e M$.

2. RECOLLECTION OF SCHUR FUNCTORS AND SCHUR COMPLEXES

In this section, we recall the definition of Schur complexes defined in [ABW]. As vector spaces, these are canonically isomorphic to images of the underlying vector spaces of the Schur superfunctors to be defined in the next section. We also recall the definitions of Schur and Weyl functors, which are fundamental examples of objects in categories of *strict polynomial functors* (cf. [Tou], [Kr]).

2.1. Strict polynomial functors and bifunctors. Let $\text{vec} = \text{vec}_{\mathbb{k}}$ denote the category of finite dimensional \mathbb{k} -vector spaces. Then define the category $\mathcal{D}_{\mathbb{k}}^d = \mathcal{D}^d(\text{vec})$ with the same objects as vec and with morphisms

$$\text{hom}_{\mathcal{D}_{\mathbb{k}}^d}(V, W) := D^d \text{Hom}(V, W),$$

for $V, W \in \text{vec}$, where $D^d(\)$ denotes the ordinary d -th divided power of a vector space. The composition of morphisms in $\mathcal{D}_{\mathbb{k}}^d$ is given by the isomorphisms

$$D^d(\text{Hom}(V, W)) = (\text{Hom}(V, W)^{\otimes d})^{\mathfrak{S}_d} \cong \text{Hom}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d} \cong \text{Hom}_{\mathbb{k}\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}),$$

where \mathfrak{S}_d acts on $V^{\otimes d}, W^{\otimes d}$ by place permutations, and on $\text{Hom}(V^{\otimes d}, W^{\otimes d})$ by conjugation.

Notice that $\mathcal{D}_{\mathbb{k}}^d$ is a \mathbb{k} -linear category; i.e. hom-sets are vector spaces, and the composition is bilinear. The category of (*homogeneous*) *strict polynomial functors* is defined to be

$$\mathcal{P}_d = \text{Fct}_{\mathbb{k}}(\mathcal{D}_{\mathbb{k}}^d, \text{vec}),$$

the category consisting of all \mathbb{k} -linear functors $T : \mathcal{D}_{\mathbb{k}}^d \rightarrow \text{vec}$. Hence if $T \in \mathcal{P}_d$, then for any $V, W \in \mathcal{D}_{\mathbb{k}}^d$, the induced map

$$T_{V,W} : \text{hom}_{\mathcal{D}_{\mathbb{k}}^d}(V, W) \rightarrow \text{Hom}(TV, TW)$$

is linear. Morphisms in \mathcal{P}_d are natural transformations between functors.

The category \mathcal{P}_d is abelian, with the existence of kernels, cokernels, products and coproducts being induced from the target category vec . Typical examples of objects in \mathcal{P}_d are respectively given by the exterior power, Λ^d , the symmetric power, Sym^d , and the divided power, D^d , of a vector space.

We now define tensor products of strict polynomial functors following [Kr, Sec. 2]. Let us define a category $\mathcal{D}_{\mathbb{k}}^d \otimes \mathcal{D}_{\mathbb{k}}^e$ with the same objects as vec and with morphisms

$$\text{hom}_{\mathcal{D}_{\mathbb{k}}^d \otimes \mathcal{D}_{\mathbb{k}}^e}(V, W) = \text{hom}_{\mathcal{D}_{\mathbb{k}}^d}(V, W) \otimes \text{hom}_{\mathcal{D}_{\mathbb{k}}^e}(V, W).$$

Given nonnegative integers d and e , we have an embedding $\mathfrak{S}_d \times \mathfrak{S}_e \hookrightarrow \mathfrak{S}_{d+e}$. This induces an embedding

$$D^{d+e}V \hookrightarrow D^dV \otimes D^eV, \tag{11}$$

for any $V \in \text{vec}$, given by the composition of the following maps

$$D^{d+e}V = (V^{\otimes d+e})^{\mathfrak{S}_{d+e}} \subset (V^{\otimes d+e})^{\mathfrak{S}_d \times \mathfrak{S}_e} \simeq (V^{\otimes d})^{\mathfrak{S}_d} \otimes (V^{\otimes e})^{\mathfrak{S}_e} = \Gamma^d V \otimes \Gamma^e V.$$

We then have a \mathbb{k} -linear functor

$$i_{d,e} : \mathcal{D}_{\mathbb{k}}^{d+e} \hookrightarrow \mathcal{D}_{\mathbb{k}}^d \otimes \mathcal{D}_{\mathbb{k}}^e,$$

which is the identity on objects and which acts by (11) on morphisms. Composition with this functor gives a tensor product $-\otimes- : \mathcal{P}_d \times \mathcal{P}_e \rightarrow \mathcal{P}_{d+e}$. I.e., we define $(S \otimes T)(V) = S(V) \otimes T(V)$ for any $V \in \mathcal{D}_{\mathbb{k}}^{d+e}$ and $S \otimes T$ acts by $i_{d,e}$ on morphisms.

Example 2.1.1. Suppose $\alpha \in (\mathbb{Z}_{\geq 0})^n$ and $|\alpha| = \sum \alpha_i = d$. Then we have polynomial functors, $\text{Sym}^\alpha, \Lambda^\alpha, D^\alpha \in \mathcal{P}_d$, defined as the tensor products

$$X^\alpha = X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_n},$$

for $X = \text{Sym}, \Lambda, D$ respectively.

We also define strict polynomial bifunctors. For this, we first define the category $\mathcal{D}_{\mathbb{k} \times \mathbb{k}}^d = \mathcal{D}^d(\text{vec} \times \text{vec})$ with the same objects as $\text{vec} \times \text{vec}$ and with morphisms

$$\text{hom}_{\mathcal{D}_{\mathbb{k} \times \mathbb{k}}^d}((V, W), (V', W')) = \bigoplus_{e+f=d} \text{hom}_{\mathcal{D}_{\mathbb{k}}^e}(V, W) \otimes \text{hom}_{\mathcal{D}_{\mathbb{k}}^f}(V', W')$$

Then define the category of (*homogeneous*) *strict polynomial bifunctors*, $\mathcal{P}^{[2]} = \text{Fct}_{\mathbb{k}}(\mathcal{D}_{\mathbb{k} \times \mathbb{k}}^d, \text{vec})$.

Next we define an external tensor product

$$- \boxtimes - : \mathcal{P}_d \times \mathcal{P}_e \rightarrow \mathcal{P}_{d+e}^{[2]}.$$

Let $\mathcal{D}_{\mathbb{k}}^e \boxtimes \mathcal{D}_{\mathbb{k}}^f \subset \mathcal{D}_{\mathbb{k} \times \mathbb{k}}^d$ denote the subcategory with the same objects as $\mathcal{D}_{\mathbb{k} \times \mathbb{k}}^d$ but with morphisms:

$$\text{hom}_{\mathcal{D}_{\mathbb{k}}^e \boxtimes \mathcal{D}_{\mathbb{k}}^f}((V, W), (V', W')) = \text{hom}_{\mathcal{D}_{\mathbb{k}}^e}(V, W) \otimes \text{hom}_{\mathcal{D}_{\mathbb{k}}^f}(V', W').$$

Now suppose $S \in \mathcal{P}_d$, $T \in \mathcal{P}_e$. Let $S \boxtimes T \in \mathcal{P}_{d+e}^{[2]}$ denote the functor defined by setting, for $V, W \in \text{vec}$,

$$(S \boxtimes T)(V, W) := S(V) \otimes T(W),$$

and $S \otimes T$ acts on morphisms via the embedding $\mathcal{D}_{\mathbb{k}}^e \boxtimes \mathcal{D}_{\mathbb{k}}^f \hookrightarrow \mathcal{D}_{\mathbb{k} \times \mathbb{k}}^d$

2.2. Schur and Weyl functors. Now let \mathbb{Z}_+^∞ denote the set of all infinite sequences $(\alpha_1, \alpha_2, \dots)$ of nonnegative integers such that $\alpha_i = 0$ for all but finitely many indices. We identify any $\alpha \in (\mathbb{Z}_{\geq 0})^n$ as an element of \mathbb{Z}_+^∞ by setting $\alpha_i = 0$ for $i \geq n + 1$. For $\alpha \in \mathbb{Z}_+^\infty$, we write $|\alpha| = \sum \alpha_i$. Also, let $l(\alpha)$ denote the cardinality of $\{i : \alpha_i \neq 0\}$.

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{Z}_+^\infty$ with non-increasing entries: $\lambda_1 \geq \lambda_2 \geq \dots$. Let \mathcal{P} denote the set of all partitions. We identify each $\lambda \in \mathcal{P}$ with its corresponding diagram

$$\Delta_\lambda := \{(i, j) \in (\mathbb{Z}_{>0})^2 : 1 \leq j \leq \lambda_i\}.$$

A *skew partition* λ/μ is a pair of partitions such that $\mu \subset \lambda$; i.e., $\mu_i \leq \lambda_i$ for $i \geq 1$. The corresponding *skew diagram*, $\Delta_{\lambda/\mu}$, is the complement of Δ_μ in the set Δ_λ . For $\lambda \in \mathcal{P}$, let $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ denote the conjugate partition, where λ'_i equals the cardinality of $\{j : \lambda_j \geq i\}$.

Let us first define a certain permutation associated to any skew partition. Suppose $\mu \subset \lambda$ are partitions such that $|\lambda| - |\mu| = d$. Then each integer $r \in \{1, \dots, d\}$ can be written uniquely as a sum

$$r = (\lambda_1 - \mu_1) + \dots + (\lambda_{i-1} - \mu_{i-1}) + j$$

with $1 \leq j \leq \lambda_i - \mu_i$. The pair $(i, j) = (i, \mu_i + j)$ describes the position (i -th row, j -th column) of r in the skew Young diagram $\Delta_{\lambda/\mu}$, and λ/μ determines a permutation $\sigma_{\lambda/\mu} \in \mathfrak{S}_d$ by

$$\sigma_{\lambda/\mu}(r) = (\lambda'_1 - \mu'_1) + \dots + (\lambda'_{j-1} - \mu'_{j-1}) + (i - \mu'_j),$$

where $1 \leq i \leq \lambda'_j - \mu'_j$. Note that $\sigma_{\lambda'/\mu'} = \sigma_{\lambda/\mu}^{-1}$.

Example 2.2.1. Let $\lambda = (4, 3, 1)$ and $\mu = (2, 1)$. Then we may number the corresponding skew Young diagram $\Delta_{(4,3,1)/(2,1)}$ in two different ways as follows:

$$\begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 4 & \\ \hline 5 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline & 3 & 5 \\ \hline & 2 & 4 \\ \hline 1 & & \\ \hline \end{array}.$$

The first numbering is from left to right in rows and from top to bottom, while the second numbering proceeds down columns and then from left to right. One may check that sending a number in a square of the left-hand diagram to the corresponding number in the same square of the right-hand diagram gives rise to the permutation $\sigma_{\lambda/\mu}$. In this case, we have $\sigma_{(4,3,1)/(2,1)} = (1, 3, 2, 5)$, written in cycle notation.

Next we define certain natural transformations in the category \mathcal{P}_d , for $d = |\lambda/\mu|$. First, let $\theta_{\lambda/\mu}(V)$ be the composite of the linear maps

$$\Lambda^{\lambda'/\mu'} V \hookrightarrow V^{\otimes d} \xrightarrow{\rho(\sigma_{\lambda/\mu})} V^{\otimes d} \twoheadrightarrow \text{Sym}^{\lambda/\mu} V,$$

where the first map is the canonical inclusion, ρ is the representation of the symmetric group acting by ordinary place permutations, and the last map is given by d -fold multiplication in the algebra $\text{Sym} V$.

Next let $\theta'_{\lambda/\mu}$ be defined as the composite

$$D^{\lambda'/\mu'} W \hookrightarrow W^{\otimes d} \xrightarrow{\rho^-(\sigma_{\lambda'/\mu'})} W^{\otimes d} \twoheadrightarrow \Lambda^{\lambda/\mu} W,$$

Definition 2.2.2 (Schur and Weyl functors). Define the *Schur functor*, $S_{\lambda/\mu}$, of shape λ/μ , to be the image of $\theta_{\lambda/\mu}$ in the category \mathcal{P}_d . The *Weyl functor*, $W_{\lambda/\mu}$, is the image of the natural transformation $\theta'_{\lambda/\mu}$.

Remark 2.2.3. A similar construction of the Schur and Weyl functors, S_λ and W_λ , corresponding to a partition λ may be found in [Tou] or [Kr]. Note that when comparing our definition with the original definition in [ABW] one must replace λ/μ by the conjugate shape λ'/μ' . For a useful dictionary of the various notations for Schur and Weyl functors (or modules) appearing in the literature, see [Tou, Section 6.1.1].

2.3. Schur complexes. We define the following strict polynomial bifunctors

$$(\Lambda \boxtimes D)^d = \bigoplus_{i=0}^d \Lambda^i \boxtimes D^{d-i} \quad \text{and} \quad (\text{Sym} \boxtimes \Lambda)^d = \bigoplus_{i=0}^d \text{Sym}^i \boxtimes \Lambda^{d-i}.$$

Notice there is a canonical embedding $(\Lambda \boxtimes D)^d(V, W) \hookrightarrow (V \oplus W)^d$ given by tensoring the canonical embeddings for Λ^i and D^j :

$$\bigoplus_{i+j=d} \Lambda^i V \otimes D^j W \hookrightarrow \bigoplus_{i+j=d} V^{\otimes i} \otimes W^{\otimes j} \subset (V \oplus W)^d.$$

Hence there is a canonical embedding $(\Lambda \boxtimes D)^d \hookrightarrow (I^{\otimes d})^{\text{bi}}$.

Also, $\text{Sym} V \otimes \Lambda W$ is a graded algebra such that

$$(\text{Sym} V \otimes \Lambda W)_d = \bigoplus_{i=0}^d \text{Sym}^i V \otimes D^{d-i} W = (\text{Sym} \boxtimes D)^d(V, W).$$

Hence, d -fold multiplication in the algebra $\text{Sym} V \otimes DW$ yields a canonical map $(V \oplus W)^{\otimes d} \twoheadrightarrow (\text{Sym} \boxtimes D)^d(V, W)$. This yields a natural transformation $(I^{\otimes d})^{\text{bi}} \twoheadrightarrow (\text{Sym} \boxtimes \Lambda)^d$.

Next, we define an action of the symmetric group on $(V \oplus W)^{\otimes d}$ which is a combination of the two action, ρ and ρ^- , considered above. Let $\rho^{\text{sc}} : \mathfrak{S}_d \rightarrow \text{End}((V \oplus W)^{\otimes d})$ be the unique representation such that the transposition $(i \ i+1)$ maps:

$$u_1 \otimes \cdots \otimes u_d \mapsto \begin{cases} -(u_1 \otimes \cdots \otimes u_{i+1} \otimes u_i \otimes \cdots \otimes u_d) & \text{if } u_i \in W \text{ and } u_{i+1} \in W, \\ u_1 \otimes \cdots \otimes u_{i+1} \otimes u_i \otimes \cdots \otimes u_d & \text{if } u_i \in V \text{ or } u_{i+1} \in V. \end{cases}$$

Note that if we consider V, W as purely even superspaces, then this agrees with the action of \mathfrak{S}_d on the superspace $(V \oplus \Pi W)^{\otimes d}$ described in Section 1.4 above. This then defines a natural transformation $\rho^{\text{sc}}(\sigma) : (I^{\otimes d})^{\text{bi}} \rightarrow (I^{\otimes d})^{\text{bi}}$, for any $\sigma \in \mathfrak{S}_d$.

Now let $\theta_{\lambda/\mu}^{\text{sc}}$ be the natural transformation defined as the composite

$$(D \boxtimes \Lambda)^{\lambda'/\mu'} \hookrightarrow (I^{\otimes d})^{\text{bi}} \xrightarrow{\rho^{\text{sc}}(\sigma_{\lambda/\mu})} (I^{\otimes d})^{\text{bi}} \twoheadrightarrow (\text{Sym} \boxtimes \Lambda)^{\lambda/\mu},$$

where the first map is given by tensoring the canonical inclusions, and the third map is given by tensoring the respective multiplication maps.

Definition 2.3.1. The image of the natural transformation $\theta_{\lambda/\mu}^{\text{sc}}$ in $(\mathcal{P}_d)^{\text{bi}}$, denoted by $SC_{\lambda/\mu}$, is called the *Schur complex bifunctor* of shape λ/μ .

We notice that, for $V, W \in \text{vec}$, we have $\theta_{\lambda/\mu}^{sc}(V, 0) = \theta_{\lambda/\mu}(V)$ and $\theta_{\lambda/\mu}^{sc}(0, W) = \theta'_{\lambda/\mu}(W)$. Hence, we have that

$$S_{\lambda/\mu} = SC_{\lambda/\mu}(-, 0) \quad \text{and} \quad W_{\lambda/\mu} = SC_{\lambda/\mu}(0, -). \quad (12)$$

Remark 2.3.2. To any linear map $\phi : W \rightarrow V$, there was defined in [ABW] an associated complex, “ $L_{\lambda'/\mu'}\phi$,” called the *Schur complex* of shape λ/μ . It is not difficult to check that $SC_{\lambda/\mu}(V, W)$ is (linearly) isomorphic to the Schur complex $L_{\lambda'/\mu'}\phi$ of the conjugate shape, regardless of which map ϕ is chosen.

3. THE SCHUR AND WEYL SUPERFUNCTORS

In this section, we construct the Schur and Weyl superfunctors via certain Svec-enriched natural transformations in the categories of polynomial superfunctors. We also provide a standard basis for the evaluation of a Schur superfunctor on a vector superspace in terms of tableaux.

3.1. Categories of strict polynomial superfunctors. Suppose A is an associative superalgebra, and let $d \in \mathbb{Z}_{\geq 1}$. Notice that the right action of $\sigma \in \mathfrak{S}_d$ on the tensor power $A^{\otimes d}$ is in fact a superalgebra automorphism. Denote by $A \wr \mathfrak{S}_d$ the vector superspace

$$A \wr \mathfrak{S}_d = \mathbb{k}\mathfrak{S}_d \otimes A^{\otimes d}$$

(where the group algebra $\mathbb{k}\mathfrak{S}_d$ is viewed as superspace concentrated in degree zero). We then consider $A \wr \mathfrak{S}_d$ as a superalgebra with multiplication defined by the rule

$$(\sigma \otimes a)(\sigma' \otimes b) = \sigma\sigma' \otimes (a \cdot \sigma')b$$

for $\sigma, \sigma' \in \mathfrak{S}_d$, $a, b \in A$. In what follows, we will identify $A^{\otimes d}$ (resp. $\mathbb{k}\mathfrak{S}_d$) with the subsuperalgebra $1 \otimes A^{\otimes d}$ (resp. $\mathbb{k}\mathfrak{S}_d \otimes 1$) of $A \wr \mathfrak{S}_d$.

Example 3.1.1 (Sergeev superalgebra). If $A = \mathbb{k}$, then $\mathbb{k} \wr \mathfrak{S}_d = \mathbb{k}\mathfrak{S}_d$, the group algebra of \mathfrak{S}_d . On the other hand, if we identify \mathcal{C}_d with $\mathcal{C}_1^{\otimes d}$ via the isomorphism in Example 1.1.1 then $\mathcal{C}_1 \wr \mathfrak{S}_d = \mathcal{W}_d$, the *Sergeev superalgebra* (cf. [BrK1]).

Recall that a superalgebra is called *simple* if it has no nonzero proper superideals. E.g., \mathbb{k} and \mathcal{C}_1 are both simple superalgebras. Suppose now that B is a simple finite dimensional superalgebra. For each $d \in \mathbb{Z}_{\geq 0}$, we define a new category $\Gamma_B^d = \Gamma^d(\text{smod } B)$. The objects of Γ_B^d are the same as those of $\text{smod } B$, i.e. finite dimensional right B -supermodules. Given $M, N \in \text{smod } B$, set

$$\text{hom}_{\Gamma_B^d}(M, N) := \Gamma^d \text{Hom}_B(V, W).$$

In order to define a composition law, we need the following:

Lemma 3.1.2 ([Ax], Lemma 3.1). *Let $V \in \text{smod } B$. Then $V^{\otimes d} \in \text{smod } B \wr \mathfrak{S}_d$. Furthermore if $V, W \in \text{smod } B$, then we have a natural isomorphism*

$$\text{Hom}_{B \wr \mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}) \simeq \Gamma^d \text{Hom}_B(V, W). \quad (13)$$

Using the isomorphism (13) for any $V, W \in \text{smod } B$, composition in $\text{smod } B \wr \mathfrak{S}_d$ then induces a composition law in Γ_B^d . In particular, we are concerned with the categories

$$\Gamma_{\mathbb{M}}^d = \Gamma_{\mathbb{k}}^d \quad \text{and} \quad \Gamma_{\mathbb{Q}}^d = \Gamma_{\mathcal{C}_1}^d.$$

This notation comes from the fact that $\mathbb{k} \cong \mathcal{M}_{1|0}$ and $\mathcal{C}_1 \cong \mathcal{Q}_1$ are the smallest examples, respectively, in the series $\mathcal{M}_{m|n}$, \mathcal{Q}_n of all finite dimensional simple superalgebras. (See Example 2.4 and Remark 2.8 of [Ax] for more details.)

Clearly, $\Gamma_{\mathbb{M}}^d$ and $\Gamma_{\mathbb{Q}}^d$ are both Svec-enriched categories.

Definition 3.1.3. Let $\text{Pol}_d^{\text{I}} = \text{Fct}_{\mathbb{k}}(\Gamma_{\mathbb{M}}^d, \text{svec})$, the category of all even \mathbb{k} -linear functors from $\Gamma_{\mathbb{M}}^d$ to svec . Similarly, let $\text{Pol}_d^{\text{II}} = \text{Fct}_{\mathbb{k}}(\Gamma_{\mathbb{Q}}^d, \text{svec})$. In both cases, morphisms are Svec-enriched natural transformations (see Section 1.1). The objects of $\text{Pol}_d^{\text{I}}, \text{Pol}_d^{\text{II}}$ are called (*homogeneous, strict*) *polynomial superfunctors of type I, II*, respectively.

Notice that Pol_d^I and Pol_d^{II} are both Svec -enriched categories, and the underlying even subcategories $(\text{Pol}_d^I)_{\text{ev}}$ and $(\text{Pol}_d^{II})_{\text{ev}}$ are abelian. In [Ax, Section 4.2], the objects of Pol_d^{II} are also referred to as *spin polynomial functors*.

Example 3.1.4. In the category Pol_1^I , we have the identity $I : \text{svec} \rightarrow \text{svec}$. The (right) parity change functor, $\Pi : \text{svec} \rightarrow \text{svec}$, is also seen to be an object of Pol_1^I . Notice that we have an odd Svec -enriched natural isomorphism $id : I \rightarrow \Pi$, where $id_M : M \rightarrow \Pi M$ is just the identity map on M .

The functor, $F_C : \text{smod } \mathcal{C}_1 \rightarrow \text{svec}$, which forgets the \mathcal{C}_1 -action, induces a forgetful functor $F_{C,d} : \Gamma_{\mathbb{Q}}^d \rightarrow \Gamma_{\mathbb{M}}^d$, for any $d \geq 0$. Composition with $F_{C,d}$ then induces a functor

$$\text{Res}_{C,d} = - \circ F_{C,d} : \text{Pol}_d^I \rightarrow \text{Pol}_d^{II}.$$

We again let Π denote the functor $\text{Res}_{C,1}(\Pi) \in \text{Pol}_1^{II}$.

Example 3.1.5. Let $d \in \mathbb{Z}_{\geq 0}$, and suppose $V \in \Gamma_{\mathbb{X}}^d$ ($\mathbb{X} = \mathbb{M}, \mathbb{Q}$). Then we define the following objects of $\text{Pol}_d^I, \text{Pol}_d^{II}$:

$$\Gamma^{d,V} := \text{hom}_{\Gamma_{\mathbb{X}}^d}(V, \cdot), \quad S_V^d := \text{hom}_{\Gamma_{\mathbb{X}}^d}(\cdot, V^*)^*.$$

Let us set $\Gamma^d = \Gamma^{d,\mathbb{K}}$ and $\Gamma_{\Pi}^d = \Gamma^{d,\mathbb{K}^{0|1}}$ in Pol_d^I , while in the category Pol_d^{II} we write $\Gamma^d = \Gamma^{d,\mathcal{C}_1}$ and $\Gamma_{\Pi}^d = \Gamma^{d,\Pi\mathcal{C}_1}$. There is clearly an even isomorphism $\Gamma^d \simeq \Gamma_{\Pi}^d$ of objects in Pol_d^{II} , since $\Pi\mathcal{C}_1 \simeq \mathcal{C}_1$ in $\Gamma_{\mathbb{Q}}^d$. Notice that $\Gamma^1 = I$, $\Gamma_{\Pi}^1 = \Pi$ in both Pol_d^I and Pol_d^{II} . The objects S^d and S_{Π}^d in $\text{Pol}_d^I, \text{Pol}_d^{II}$ are defined in a similar way.

We recall that the *Kuhn dual* of $T \in \text{Pol}_d^{\dagger}$ ($\dagger = I, II$) is the object $T^{\#} \in \text{Pol}_d^{\dagger}$ which is defined so that $T^{\#}(V) = T(V^*)^*$ for all $V \in \Gamma_{\mathbb{X}}^d$ ($\mathbb{X} = \mathbb{M}, \mathbb{Q}$). Clearly, we have a canonical isomorphism

$$S^d \simeq (\Gamma^d)^{\#} \quad (14)$$

of objects in Pol_d^{\dagger} .

We may also form the tensor product of polynomial superfunctors. Suppose given nonnegative integers d and e . Then, as in (11), the embedding $\mathfrak{S}_d \times \mathfrak{S}_e \hookrightarrow \mathfrak{S}_{d+e}$ induces an embedding

$$\Gamma^{d+e} M \hookrightarrow \Gamma^d M \otimes \Gamma^e M \quad (15)$$

for any $M \in \text{svec}$. We may also consider the categories $\Gamma_{\mathbb{M}}^d \otimes \Gamma_{\mathbb{M}}^e, \Gamma_{\mathbb{Q}}^d \otimes \Gamma_{\mathbb{Q}}^e$ whose objects are the same as $\text{svec}_{\mathbb{K}}, \text{smod } \mathcal{C}_1$ and whose morphisms are of the form

$$\text{hom}_{\Gamma_{\mathbb{M}}^d}(M, N) \otimes \text{hom}_{\Gamma_{\mathbb{M}}^e}(M, N), \quad \text{hom}_{\Gamma_{\mathbb{Q}}^d}(V, W) \otimes \text{hom}_{\Gamma_{\mathbb{Q}}^e}(V, W)$$

respectively for $M, N \in \text{svec}_{\mathbb{K}}$ and $V, W \in \text{smod } \mathcal{C}_1$. Here, we assume that the composition of morphisms obeys the usual rule of signs conventions as in (1).

One may show that (15) yields embeddings of categories

$$\Gamma_{\mathbb{M}}^{d+e} \hookrightarrow \Gamma_{\mathbb{M}}^d \otimes \Gamma_{\mathbb{M}}^e, \quad \Gamma_{\mathbb{Q}}^{d+e} \hookrightarrow \Gamma_{\mathbb{Q}}^d \otimes \Gamma_{\mathbb{Q}}^e. \quad (16)$$

Then, as in Section 2.1, (16) yields induced bifunctors:

$$- \otimes - : \text{Pol}_d^{\dagger} \times \text{Pol}_e^{\dagger} \rightarrow \text{Pol}_{d+e}^{\dagger}.$$

For example, if we let $\alpha \in (\mathbb{Z}_{\geq 0})^n$, then we have objects

$$S^{\alpha} = S^{\alpha_1} \otimes \dots \otimes S^{\alpha_n}, \quad \Gamma^{\alpha} = \Gamma^{\alpha_1} \otimes \dots \otimes \Gamma^{\alpha_n}$$

which belong to Pol_d^{\dagger} ($\dagger = I, II$), where $d = |\alpha|$. Also, we have the objects $I^{\otimes d} = I \otimes \dots \otimes I$ and $\Pi^{\otimes d} = \Pi \otimes \dots \otimes \Pi$ which belong to Pol_d^I . Since $1 : \Pi \rightarrow I$ is an odd Svec -enriched natural transformation, we have an isomorphism

$$1^{\otimes d} : \Pi^{\otimes d} \cong I^{\otimes d} \quad (17)$$

which is an odd (resp. even) morphism if d is odd (resp. even).

Notice that the canonical embedding $\Delta : \Gamma^d M \hookrightarrow M^{\otimes d}$ and surjection $m : M^{\otimes d} \twoheadrightarrow S^d M$ of superspaces induce even morphisms

$$\Delta : \Gamma^d \hookrightarrow I^{\otimes d} \quad \text{and} \quad m : I^{\otimes d} \twoheadrightarrow S^d, \quad (18)$$

respectively. It follows from (17) and (18) that we have even morphisms:

$$\Delta_\Pi : \Gamma_\Pi^d \hookrightarrow I^{\otimes d} \quad \text{and} \quad m_\Pi : I^{\otimes d} \twoheadrightarrow S_\Pi^d, \quad (19)$$

where

$$\Delta_\Pi = \begin{cases} \Delta & \text{if } d \text{ is even,} \\ \Pi \circ \Delta & \text{if } d \text{ is odd,} \end{cases} \quad \text{and} \quad m_\Pi = \begin{cases} m & \text{if } d \text{ is even,} \\ \Pi \circ m & \text{if } d \text{ is odd.} \end{cases}$$

Also, for any $\alpha \in (\mathbb{Z}_{\geq 0})^n$, set

$$\Gamma_\Pi^\alpha := \Gamma_\Pi^{\alpha_1} \otimes \cdots \otimes \Gamma_\Pi^{\alpha_n}, \quad S_\Pi^\alpha := S_\Pi^{\alpha_1} \otimes \cdots \otimes S_\Pi^{\alpha_n}.$$

Then, by some more abuse of notation, we also have canonical even morphisms: $\Delta_\Pi : \Gamma_\Pi^\alpha \hookrightarrow I^{\otimes |\alpha|}$ and $m_\Pi : I^{\otimes |\alpha|} \twoheadrightarrow S_\Pi^\alpha$, given by taking the tensor products of the canonical maps (19).

3.2. Schur and Weyl superfunctors. We now proceed to construct super analogues of the Schur and Weyl functors. As we will see in Proposition 3.2.4 below, the image of a Schur superfunctor is canonically isomorphic to a corresponding Schur complex.

Suppose now that $\mu \subset \lambda$ are partitions and $|\lambda| - |\mu| = d$. Let us write $X^{\lambda/\mu} = X^{\lambda-\mu}$, for $X = \Gamma, S$, respectively. We then define an Svec-enriched natural transformation $\hat{\theta}_{\lambda/\mu} : \Gamma_\Pi^{\lambda'/\mu'} \rightarrow S^{\lambda/\mu}$ as the composition of morphisms

$$\Gamma_\Pi^{\lambda'/\mu'} \xrightarrow{\Delta_\Pi} I^{\otimes d} \xrightarrow{\sigma_{\lambda/\mu}} I^{\otimes d} \xrightarrow{m} S^{\lambda/\mu},$$

where the action of $\sigma_{\lambda/\mu}$ on $I^{\otimes d}$ is induced by the action of \mathfrak{S}_d on the superspace $M^{\otimes d}$, for any $M \in \Gamma_{\mathbf{X}}^d$ ($\mathbf{X} = \mathbf{M}, \mathbf{Q}$), respectively.

Recall the restriction functor, $\text{Res}_{\mathcal{C},d} : \text{Pol}_d^{\mathbf{I}} \rightarrow \text{Pol}_d^{\mathbf{II}}$, induced by the forgetful functor, $F_{\mathcal{C},d} : \Gamma_{\mathbf{Q}}^d \rightarrow \Gamma_{\mathbf{M}}^d$, described above.

Definition 3.2.1. The *Schur superfunctor*, $\hat{S}_{\lambda/\mu} = \hat{S}_{\lambda/\mu}^{\mathbf{I}}$, of *type I* is the polynomial superfunctor defined by setting $\hat{S}_{\lambda/\mu}(M)$ equal to the image of the map $\hat{\theta}_{\lambda/\mu}(M)$ for any $M \in \Gamma_{\mathbf{M}}^d$. The *Schur superfunctor of type II* is defined by restriction: $\hat{S}_{\lambda/\mu}^{\mathbf{II}} = \text{Res}_{\mathcal{C},d}(\hat{S}_{\lambda/\mu})$.

We may also define an Svec-enriched natural transformation $\check{\theta}_{\lambda/\mu} : \Gamma^{\lambda'/\mu'} \rightarrow S_\Pi^{\lambda/\mu}$ as the composition

$$\Gamma^{\lambda'/\mu'} \xrightarrow{\Delta} I^{\otimes d} \xrightarrow{\sigma_{\lambda'/\mu'}} I^{\otimes d} \xrightarrow{m_\Pi} S_\Pi^{\lambda/\mu},$$

where each map is defined in a similar way to the above.

Definition 3.2.2. The *Weyl (or co-Schur) superfunctor*, $\widehat{W}_{\lambda/\mu} = \widehat{W}_{\lambda/\mu}^{\mathbf{I}}$, of *type I* is defined by setting $\widehat{W}_{\lambda/\mu}^{\mathbf{I}}(M)$ equal to the image of $\check{\theta}_{\lambda/\mu}(M)$, and the *Weyl superfunctor of type II* is given by: $\widehat{W}_{\lambda/\mu}^{\mathbf{II}} = \text{Res}_{\mathcal{C},d}(\widehat{W}_{\lambda/\mu})$.

Remark 3.2.3. Notice that the polynomial superfunctors Π and I are both canonically isomorphic to their own Kuhn duals. It is also not difficult to check that $(S \otimes T)^\#$ is canonically isomorphic to $S^\# \otimes T^\#$ for all $S, T \in \text{Pol}_{\mathbf{k}}^{\mathbf{I}}$. It then follows from the definition of the morphisms $\hat{\theta}_{\lambda/\mu}$ and $\check{\theta}_{\lambda/\mu}$ and from (14) that there is a canonical isomorphism

$$\widehat{W}_{\lambda'/\mu'} \simeq (\hat{S}_{\lambda/\mu})^\#$$

which is analogous to the usual duality between Schur and Weyl functors (cf. [ABW, Prop. II.4.1]). For this reason, we consider only Schur superfunctors in the remainder.

Proposition 3.2.4. Suppose $M \in \Gamma_M^d$. Then there is a canonical isomorphism

$$SC_{\lambda/\mu}(\underline{M_0}, \underline{M_1}) \cong \underline{\hat{S}_{\lambda/\mu}(M)}$$

of vector spaces. Furthermore, there are natural isomorphisms

$$\hat{S}_{\lambda/\mu}(M_0) \cong S_{\lambda/\mu}(\underline{M_0}) \quad \text{and} \quad \hat{S}_{\lambda/\mu}(M_1) \cong W_{\lambda/\mu}(\underline{M_1}),$$

where $S_{\lambda/\mu}$, $W_{\lambda/\mu}$ are the ordinary Schur and Weyl functors respectively.

Proof. The first statement follows by comparing the definitions of the natural transformations $\theta_{\lambda/\mu}^{\text{sc}}$ and $\hat{\theta}_{\lambda/\mu}$. The second part then follows from the isomorphisms given in (12) above. \square

3.3. The Standard Basis Theorem. Let us introduce the relevant tableaux with which we shall need to work. Let \mathcal{Z} be a \mathbb{Z}_2 -graded set $\mathcal{Z} = \mathcal{Z}_0 \sqcup \mathcal{Z}_1$. Assume \mathcal{Z} has a total order, $<$. A *tableaux*, \mathbf{t} , of shape λ/μ with values in \mathcal{Z} is a function $\mathbf{t} : \Delta_{\lambda/\mu} \rightarrow \mathcal{Z}$. Let $\text{Tab}_{\mathcal{Z}}(\lambda/\mu)$ denote the set of all \mathcal{Z} -valued tableaux.

Definition 3.3.1. Suppose $\mathbf{t} \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$.

1. The tableau \mathbf{t} is called *row standard* (resp. *row costandard*) if the entries in each row are nondecreasing and repeated entries, if any, occur only among values in \mathcal{Z}_0 (resp. \mathcal{Z}_1).
2. It is called *column standard* (resp. *column costandard*) if the columns are nondecreasing and any repeated entries occur among the \mathcal{Z}_1 (resp. \mathcal{Z}_0).
3. The tableau \mathbf{t} is called *standard* if it is both row standard and column standard, and it is called *costandard* if it is both row costandard and column costandard.

Suppose $M \in \text{svec}$, and let $\mathcal{X} = (X_1, \dots, X_m)$ (resp. $\mathcal{Y} = (Y_1, \dots, Y_n)$) be an ordered basis of M_0 (resp. M_1). Then we fix some total order on the union $\mathcal{Z} = \mathcal{X} \sqcup \mathcal{Y} = (Z_1, \dots, Z_{m+n})$ which preserves the respective orderings in \mathcal{X} and \mathcal{Y} . This gives a \mathbb{Z}_2 -graded set with $\mathcal{Z}_0 = \mathcal{X}$, $\mathcal{Z}_1 = \mathcal{Y}$. Now, since we are dealing the Schur superfunctors, we will also need to consider the vector superspace ΠM with basis $\mathcal{Z}_\pi = \{\Pi Z_1, \dots, \Pi Z_{m+n}\}$, which is again totally ordered by the indices. Then \mathcal{Z}_π is a \mathbb{Z}_2 -graded set with $(\mathcal{Z}_\pi)_0 = \{\Pi Y_1, \dots, \Pi Y_n\}$ and $(\mathcal{Z}_\pi)_1 = \{\Pi X_1, \dots, \Pi X_m\}$.

In working with tableaux, it will be more convenient to deal with an indexing set in place of the basis \mathcal{Z} . Let us consider the set $[m+n] = [m+n]_{\mathcal{Z}} = \{1, \dots, m+n\}$ as a \mathbb{Z}_2 -graded set with $[m+n]_{\varepsilon} = \{1 \leq k \leq m+n : |Z_k| = \varepsilon\}$, for $\varepsilon \in \mathbb{Z}_2$, and with the usual total order on integers. It is clear that $\text{Tab}_{[m+n]}(\lambda/\mu)$ is in bijection with $\text{Tab}_{\mathcal{Z}}(\lambda/\mu)$, and this bijection preserves row (co)standardness and column (co)standardness of tableaux.

For any $\mathbf{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$, let $w(\mathbf{t}) \in I(m|n, d)$ denote the *reading word* of \mathbf{t} obtained by reading entries from left to right and top to bottom. I.e., $w(\mathbf{t}) = (w_1, w_2, \dots, w_d)$ with $w_1 = \mathbf{t}(1, \mu_1 + 1)$, $w_2 = \mathbf{t}(1, \mu_1 + 2)$, \dots , $w_d = \mathbf{t}(q, \lambda_q)$, where $q = l(\lambda)$. Note that this gives a bijection

$$w(\cdot) : \text{Tab}_{[m+n]}(\lambda/\mu) \xrightarrow{\sim} I(m|n, d)$$

if $d = |\lambda/\mu|$.

Now fix a tableau $\mathbf{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$. Recalling the notation of Section 1.4, we introduce the following notation for elements

$$Z^{(\mathbf{t}^i)} = Z^{(w(\mathbf{t}^i))} \in \Gamma^{\lambda_i - \mu_i} M \quad \text{and} \quad Z^{\mathbf{t}^i} = Z^{w(\mathbf{t}^i)} \in M^{\otimes \lambda_i - \mu_i}.$$

If $q = l(\lambda)$ and $d = |\lambda/\mu|$, then we further write

$$Z^{(\mathbf{t})} = Z^{(\mathbf{t}^1)} \otimes \dots \otimes Z^{(\mathbf{t}^q)} \in \Gamma^{\lambda/\mu} \quad \text{and} \quad Z^{\mathbf{t}} = Z^{\mathbf{t}^1} \otimes \dots \otimes Z^{\mathbf{t}^q} \in M^{\otimes d}.$$

Now we may also use tableaux in $\text{Tab}_{[m+n]}(\lambda/\mu)$ to parametrize elements of $\Gamma_{\Pi}^{\lambda/\mu} M$ and $(\Pi M)^{\otimes d}$. I.e., we set

$$Z_{\Pi}^{(\mathbf{t}^i)} = (\Pi Z)^{(w(\mathbf{t}^i))} \in \Gamma_{\Pi}^{\lambda_i - \mu_i} M \quad \text{and} \quad Z_{\Pi}^{\mathbf{t}^i} = (\Pi Z)^{w(\mathbf{t}^i)} \in (\Pi M)^{\otimes \lambda_i - \mu_i},$$

and we further write

$$Z_{\Pi}^{\mathbf{t}} = Z_{\Pi}^{\mathbf{t}^1} \otimes \dots \otimes Z_{\Pi}^{\mathbf{t}^q} \in (\Pi M)^{\otimes d}, \quad Z_{\Pi}^{(\mathbf{t})} = Z_{\Pi}^{(\mathbf{t}^1)} \otimes \dots \otimes Z_{\Pi}^{(\mathbf{t}^q)} \in \Gamma_{\Pi}^{\lambda/\mu}.$$

It follows from Section 1.4 that the set of $Z^{(\mathbf{t})}$ such that $\mathbf{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$ is row standard (resp. row costandard) forms a basis of $\Gamma^{\lambda/\mu}M$ (resp. $\Gamma_{\Pi}^{\lambda/\mu}M$). Also, it is clear that the sets $\{Z^{\mathbf{t}}\}$ and $\{Z_{\Pi}^{\mathbf{t}}\}$, such that $\mathbf{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$, give bases for $M^{\otimes d}$ and $(\Pi M)^{\otimes d}$, respectively.

Definition 3.3.2. Let \mathcal{Z} be a \mathbb{Z}_2 -graded set with a total order, $z_1 < \dots < z_r$, of its elements. Suppose $\mathbf{t} \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$, and let p, q be positive integers. Define $\mathbf{t}_{p,q}$ to be the number of times the elements z_1, \dots, z_q appear as entries in the first p rows of \mathbf{t} . More precisely, $\mathbf{t}_{p,q}$ is equal to the cardinality of $\{(i, j) \in \Delta_{\lambda/\mu} : 1 \leq i \leq p \text{ and } \mathbf{t}(i, j) \in \{z_1, \dots, z_q\}\}$. If \mathbf{s} is another tableau, we say $\mathbf{s} \leq \mathbf{t}$ if $\mathbf{s}_{p,q} \geq \mathbf{t}_{p,q}$ for all p, q . We say $\mathbf{s} \triangleleft \mathbf{t}$ if $\mathbf{s} \leq \mathbf{t}$ and $\mathbf{s}_{p,q} > \mathbf{t}_{p,q}$ for at least one pair p, q .

Remark 3.3.3. The above puts a quasi-order on the set $\text{Tab}_{\mathcal{Z}}(\lambda/\mu)$: \leq is reflexive and transitive, but we may have $\mathbf{t} \leq \mathbf{s}$ and $\mathbf{s} \leq \mathbf{t}$ with $\mathbf{s} \neq \mathbf{t}$. We do however obtain a genuine partial order by restricting either to the set of all row (co)standard tableaux, or to the set of all column (co)standard tableaux, of shape λ/μ .

Notice that the definition of the quasi-order above depends only on the total order, $<$, of \mathcal{Z} and not on its \mathbb{Z}_2 -grading. The statement and proof of the following lemma are thus essentially the same as given in Lemma II.2.14 of [ABW].

Lemma 3.3.4. Let $\mathbf{t} \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$, and let \mathbf{s} be the tableau, also of shape λ/μ , formed by exchanging certain entries from the k -th row of \mathbf{t} , say $\mathbf{t}(k, h_1), \dots, \mathbf{t}(k, h_{\alpha})$, with certain entries from the $(k+1)$ -st row of \mathbf{t} , say $\mathbf{t}(k+1, l_1), \dots, \mathbf{t}(k+1, l_{\alpha})$, such that $\mathbf{t}(k+1, l_{\nu}) < \mathbf{t}(k, h_{\nu})$, for $\nu = 1, \dots, \alpha$. More precisely, for $(i, j) \notin \{(k, h_1), \dots, (k, h_{\alpha}), (k+1, l_1), \dots, (k+1, l_{\alpha})\}$, $\mathbf{s}(i, j) = \mathbf{t}(i, j)$, but $\mathbf{s}(k, h_{\nu}) = \mathbf{t}(k+1, l_{\nu})$ and $\mathbf{s}(k+1, l_{\nu}) = \mathbf{t}(k, h_{\nu})$, for $\nu = 1, \dots, \alpha$. Then $\mathbf{s} \triangleleft \mathbf{t}$.

Now suppose $\alpha^1, \dots, \alpha^r \in \mathbb{Z}_+^{\infty}$, and let $\alpha = \sum \alpha^i$, so that $\alpha_j = \alpha_j^1 + \dots + \alpha_j^r$ for $j = 1, \dots, r$. Then there is a canonical even morphism

$$S^{\alpha^1} \otimes S^{\alpha^2} \otimes \dots \otimes S^{\alpha^r} \twoheadrightarrow S^{\alpha}, \quad (20)$$

which is given by first applying iterations of the super twists followed by iterations of the multiplication.

Lemma 3.3.5. The natural transformation $\hat{\theta}_{\lambda'/\mu'}$ can be factored as follows:

$$\begin{aligned} & \Gamma_{\Pi}^{\lambda_1 - \mu_1} \otimes \dots \otimes \Gamma_{\Pi}^{\lambda_i - \mu_i} \otimes \Gamma_{\Pi}^{\lambda_{i+1} - \mu_{i+1}} \otimes \dots \otimes \Gamma_{\Pi}^{\lambda_q - \mu_q} \\ & \xrightarrow{\alpha} \Gamma_{\Pi}^{\lambda_1 - \mu_1} \otimes \dots \otimes S^{(\lambda_i, \lambda_{i+1})' / (\mu_i, \mu_{i+1})'} \otimes \dots \otimes \Gamma_{\Pi}^{\lambda_q - \mu_q} \\ & \xrightarrow{\beta} S^{(\lambda_1, \dots, \lambda_{i-1})' / (\mu_1, \dots, \mu_{i-1})'} \otimes S^{(\lambda_i, \lambda_{i+1})' / (\mu_i, \mu_{i+1})'} \otimes S^{(\lambda_{i+2}, \dots, \lambda_q)' / (\mu_{i+2}, \dots, \mu_q)'} \\ & \xrightarrow{\gamma} S^{\lambda'/\mu'}, \end{aligned}$$

where α is the map $1 \otimes \dots \otimes \hat{\theta}_{(\lambda_i, \lambda_{i+1}) / (\mu_i, \mu_{i+1})} \otimes \dots \otimes 1$, β is the map $\hat{\theta}_{(\lambda_1, \dots, \lambda_{i-1}) / (\mu_1, \dots, \mu_{i-1})} \otimes 1 \otimes \hat{\theta}_{(\lambda_{i+2}, \dots, \lambda_n) / (\mu_{i+2}, \dots, \mu_n)}$, and γ is the map

$$S^{\alpha^1} \otimes S^{\alpha^2} \otimes S^{\alpha^3} \twoheadrightarrow S^{\lambda'/\mu'}$$

given by (20) with $\alpha^1 = (\lambda_1, \dots, \lambda_{i-1})' - (\mu_1, \dots, \mu_{i-1})'$, $\alpha^2 = (\lambda_i, \lambda_{i+1})' - (\mu_i, \mu_{i+1})'$ and $\alpha^3 = (\lambda_{i+2}, \dots, \lambda_n)' - (\mu_{i+2}, \dots, \mu_n)'$.

We wish to define a natural transformation, $\diamond_{\lambda/\mu}$, whose image lies in $\Gamma_{\Pi}^{\lambda/\mu}$ and such that

$$\hat{S}_{\lambda'/\mu'} \simeq \Gamma_{\Pi}^{\lambda/\mu} / \text{Im} \diamond_{\lambda/\mu}.$$

By the above lemma, we may reduce to the case where $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$. Now if $\mu_1 \geq \lambda_2$, then $S^{\lambda'/\mu'} \cong M^{\otimes d}$, where $d = \lambda_1 + \lambda_2 - \mu_1 - \mu_2$. In this case, $\hat{\theta}_{\lambda/\mu} : \Gamma_{\Pi}^{\lambda/\mu} \rightarrow M^{\otimes d}$

is just the canonical embedding, and we have an isomorphism $\widehat{S}_{\lambda/\mu} \cong \Gamma_{\Pi}^{\lambda/\mu}$. Hence, we may further reduce to the case where $\mu = (\mu_1, \mu_2) \subset (\lambda_1, \lambda_2) = \lambda$, with $\mu_1 < \lambda_2$.

Definition 3.3.6. Let $M \in \text{svec}$, and suppose $\mu \subset \lambda$ are arbitrary partitions with $l(\lambda) = q$. Suppose also that $1 \leq i \leq q-1$, and let u, v be nonnegative integers such that $u+v < \lambda_{i+1} - \mu_i$. Then we define a map $\diamond_i(\lambda/\mu, u, v; M)$ as the composition

$$\begin{aligned} (\Gamma_{\Pi}^u \otimes \Gamma_{\Pi}^{p_1-u+p_2-v} \otimes \Gamma_{\Pi}^v)(M) &\xrightarrow{1 \otimes \Delta \otimes 1} (\Gamma_{\Pi}^u \otimes \Gamma_{\Pi}^{p_1-u} \otimes \Gamma_{\Pi}^{p_2-v} \otimes \Gamma_{\Pi}^v)(M) \\ &\xrightarrow{m \otimes m} (\Gamma_{\Pi}^{p_1} \otimes \Gamma_{\Pi}^{p_2})(M), \end{aligned}$$

where $p_1 = \lambda_i - \mu_i$, $p_2 = \lambda_{i+1} - \mu_{i+1}$ and Δ is the appropriate component of the comultiplication in $\Gamma(\Pi M)$, while m denotes multiplication. We then define the map $\diamond_{\lambda/\mu}(M)$ to be the sum of all these maps, i.e.,

$$\diamond_{\lambda/\mu}(M) = \sum_{i=1}^{q-1} \sum_{u,v} 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \diamond_i(\lambda/\mu, u, v; M) \otimes 1_{i+2} \cdots \otimes 1_n,$$

where the inner sum is over pairs u, v such that $0 \leq u+v < \lambda_{i+1} - \mu_i$. This defines an Svec -enriched natural transformation, $\diamond_{\lambda/\mu}$, whose image lies in $\Gamma_{\Pi}^{\lambda/\mu}$.

Suppose now that $M \in \text{svec}$ with ordered basis $\mathcal{Z} = \mathcal{Z}_0 \sqcup \mathcal{Z}_1 = (Z_1, \dots, Z_{m+n})$ as above. We have the following.

Lemma 3.3.7 (Straightening Law). *Suppose that $\mathbf{t} \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$ is row costandard, but not costandard. Then there are row costandard tableaux $\mathbf{t}_l \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$ such that $\mathbf{t}_l \neq \mathbf{t}$ with $\mathbf{t}_l \triangleleft \mathbf{t}$ and $Z_{\Pi}^{(\mathbf{t})} + \sum_l C_l Z_{\Pi}^{(\mathbf{t}_l)} \in \text{Im } \diamond_{\lambda/\mu}(M)$ for some integers C_l .*

Proof. Suppose first that $l(\lambda) = 2$. Then there is a column j such that either:

$$\mathbf{t}(1, j) > \mathbf{t}(2, j), \quad \text{or} \quad \mathbf{t}(1, j), \mathbf{t}(2, j) \in [m+n]_1 \text{ and } \mathbf{t}(1, j) = \mathbf{t}(2, j).$$

Let j_0 denote the minimal such j with this property.

Now set $u = j_0 - \mu_1 - 1$ and $v = \lambda_2 - j_0$. Let us also write $p_i = \lambda_i - \mu_i$ for $i = 1, 2$. If we let g_1, \dots, g_{p_1} (resp. h_1, \dots, h_{p_2}) denote the entries in the first (resp. second) row, then the tableau \mathbf{t} has the form

	g_1	\cdots	g_u	$g_{u+1} = g_{j_0-\mu_1}$	\cdots	g_{p_1}
h_1	\cdots		$h_{j_0-\mu_2} = h_{p_2-v}$	h_{p_2-v+1}	\cdots	h_{p_2-v+r}

where r is the largest nonnegative integer such that: $h_{p_2-v} = \cdots = h_{p_2-v+r}$. Recall from Section 1.4 that $\Gamma_{\Pi}^u(M) \otimes \Gamma_{\Pi}^{p_1-u+p_2-v+r}(M) \otimes \Gamma_{\Pi}^{v-r}(M)$ has a basis of elements of the form

$$Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})} = Z_{\Pi}^{(\mathbf{a})} \otimes Z_{\Pi}^{(\mathbf{b})} \otimes Z_{\Pi}^{(\mathbf{c})},$$

for standardized sequences $\mathbf{a}, \mathbf{b}, \mathbf{c} \in I(m|n)$ such that $l(\mathbf{a}) = u$, $l(\mathbf{b}) = p_1 + p_2 - u - v + r$, and $l(\mathbf{c}) = v - r$. Let us fix the sequences $\mathbf{a}, \mathbf{b}, \mathbf{c}$ determined by setting

$$a_j = \mathbf{t}(1, j), \quad \mu_1 + 1 \leq j < j_0; \quad c_j = \mathbf{t}(2, j), \quad j_0 + r + 1 \leq \lambda_2 < j_0; \quad \text{and}$$

$$b_j = \begin{cases} \mathbf{t}(1, j), & j_0 \leq j \leq \lambda_1 \\ \mathbf{t}(2, j), & \mu_2 + 1 \leq j < j_0 + r. \end{cases}$$

The fact that \mathbf{a}, \mathbf{c} are \mathcal{Z}_{π} -restricted follows immediately from the fact that \mathbf{t} is row costandard. On the other hand, one may check that \mathbf{b} is \mathcal{Z}_{π} -restricted by using the row-costandardness of \mathbf{t} together with the fact that $h_{p_2-v-1} \leq g_u$, with $h_{p_2-v} = g_{u+1}$ only if both g_u and h_{p_2-v-1} belong to $[m+n]_1$.

We now wish to show that $\diamond_1(\lambda/\mu, u, v; M)(Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})})$ contains the basis element $Z_{\Pi}^{(\mathbf{t})}$ as a summand with coefficient 1, and all other nonzero summands correspond to row costandard tableaux occurring before \mathbf{t} in the order \triangleleft . From Lemma 1.4.1.(i), we have

$$\diamond_1(\lambda/\mu, u, v; M)(Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}) = \sum_{\text{st}_{\mathcal{Z}}(\mathbf{b}^1 \vee \mathbf{b}^2) = \mathbf{b}} Z_{\Pi}^{(\mathbf{a})} * Z_{\Pi}^{(\mathbf{b}^1)} \otimes Z_{\Pi}^{(\mathbf{b}^2)} * Z_{\Pi}^{(\mathbf{c})}$$

where the sum ranges over pairs of standardized sequences $\mathbf{b}^1, \mathbf{b}^2$. To each pair $\mathbf{b}^1, \mathbf{b}^2$, with $\text{st}_{\mathcal{Z}}(\mathbf{b}^1 \vee \mathbf{b}^2) = \mathbf{b}$, there corresponds a unique tableau $\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2)$ of shape λ/μ such that $w(\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2)) = \text{st}(\mathbf{a} \vee \mathbf{b}^1) \vee \text{st}(\mathbf{b}^2 \vee \mathbf{c})$. Hence, from Lemma 1.4.1.(ii) we have

$$Z_{\Pi}^{(\mathbf{a})} * Z_{\Pi}^{(\mathbf{b}^1)} \otimes Z_{\Pi}^{(\mathbf{b}^2)} * Z_{\Pi}^{(\mathbf{c})} = C(\mathbf{b}^1, \mathbf{b}^2) Z_{\Pi}^{(\mathbf{a} \vee \mathbf{b}^1)} \otimes Z_{\Pi}^{(\mathbf{b}^2 \vee \mathbf{c})} = \pm C(\mathbf{b}^1, \mathbf{b}^2) Z_{\Pi}^{(\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2))},$$

for some nonnegative integer $C(\mathbf{b}^1, \mathbf{b}^2)$. If $\text{st}(\mathbf{a} \vee \mathbf{b}^1)$ and $\text{st}(\mathbf{b}^2 \vee \mathbf{c})$ are both \mathcal{Z}_{π} -restricted, then $\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2)$ is row costandard and $C(\mathbf{b}^1, \mathbf{b}^2) > 0$. Otherwise, we have $Z_{\Pi}^{(\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2))} = 0$. Now there is exactly one pair $\dot{\mathbf{b}}^1, \dot{\mathbf{b}}^2$ such that $\mathbf{t} = \mathbf{t}(\dot{\mathbf{b}}^1, \dot{\mathbf{b}}^2)$. Furthermore, all of the other tableaux $\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2) \neq \mathbf{t}$ are obtained from \mathbf{t} by the type of exchange described in Lemma 3.3.4, so that $\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2) \triangleleft \mathbf{t}$.

It thus remains to show that $Z_{\Pi}^{(\mathbf{t})}$ occurs as a summand with coefficient $C(\dot{\mathbf{b}}^1, \dot{\mathbf{b}}^2) = 1$. We claim that $g_u < g_{u+1}$. Suppose not. Then $g_u = g_{u+1} \in [m+n]_1$, since \mathbf{t} is row costandard. The fact that j_0 was chosen to be minimal implies that $g_u \leq h_{p_2-v-1} \leq h_{p_2-v} \leq g_{u+1}$, with $g_u = h_{p_2-v-1}$ only if $g_u, h_{p_2-v-1} \in [m+n]_0$, a contradiction. On the other hand, it follows by the definition of r that $h_{p_2-v+r} < h_{p_2-v+r+1}$. Hence, we have that $\text{Im}(\mathbf{a}) \cap \text{Im}(\dot{\mathbf{b}}^1) = \emptyset$ and $\text{Im}(\dot{\mathbf{b}}^2) \cap \text{Im}(\mathbf{c}) = \emptyset$. It then follows from Lemma 1.4.1.(ii) that the coefficient of $Z_{\Pi}^{(\mathbf{t})}$ is 1. This completes the proof when $l(\lambda) = 2$.

Now suppose $l(\lambda) > 2$. Since \mathbf{t} is not costandard, there must be some pair $(i, j), (i+1, j) \in \triangle_{\lambda/\mu}$ such that $\mathbf{t}(i, j) > \mathbf{t}(i+1, j)$, or such that $\mathbf{t}(i, j) = \mathbf{t}(i+1, j)$ and $\mathbf{t}(i, j), \mathbf{t}(i+1, j) \in [m+n]_1$. We may then apply the above argument to the tableau $\bar{\mathbf{t}}$ which consists of i -th and $(i+1)$ -st rows of \mathbf{t} . Since the maps $\diamond_i(\lambda/\mu, u, v; M)(Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})})$ do not affect the other rows of \mathbf{t} , we may again obtain row costandard tableaux $\mathbf{t}_l \triangleleft \mathbf{t}$ such that $\mathbf{t} + \sum C_l \mathbf{t}_l \in \text{Im } \diamond_i(\lambda/\mu, u, v; M) \subset \text{Im } \diamond_{\lambda/\mu}(M)$. \square

Corollary 3.3.8. *The cosets of the elements $Z_{\Pi}^{(\mathbf{t})}$, indexed by costandard tableaux $\mathbf{t} \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$, span the superspace $\Gamma_{\Pi}^{\lambda/\mu}(M) / \text{Im } \diamond_{\lambda/\mu}(M)$.*

Proof. It is clear that the set of cosets, $Z_{\Pi}^{(\mathbf{t})} + \text{Im } \diamond_{\lambda/\mu}(M)$, corresponding to row costandard $\mathbf{t} \in \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$, generates $\Gamma_{\Pi}^{\lambda/\mu}(M) / \text{Im } \diamond_{\lambda/\mu}(M)$. The Corollary then follows from Lemma 3.3.7 by induction. \square

Proposition 3.3.9. *There is an embedding $\text{Im } \diamond_{\lambda/\mu} \hookrightarrow \text{Ker } \hat{\theta}_{\lambda'/\mu'}$ in the category $(\text{Pol}_d^1)_{\text{ev}}$.*

Proof. Let $M \in \text{svec}$. It suffices to show that $\text{Im } \diamond_{\lambda/\mu}(M) \subset \text{Ker } \hat{\theta}_{\lambda/\mu}(M)$. By Lemma 3.3.5 and the definition of $\diamond_{\lambda/\mu}(M)$, we may assume that $l(\lambda) = 2$. Then suppose we are given standardized $\mathbf{a}, \mathbf{b}, \mathbf{c} \in I(m|n)$ which are \mathcal{Z}_{π} -restricted with $l(\mathbf{a}) = u$, $l(\mathbf{b}) = p_1 + p_2 - u - v$, and $l(\mathbf{c}) = v$, where $p_1 = \lambda_1 - \mu_1$, $p_2 = \lambda_2 - \mu_2$ and $u + v < \lambda_2 - \mu_1$. Then there is a corresponding basis element

$$Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})} = Z_{\Pi}^{(\mathbf{a})} \otimes Z_{\Pi}^{(\mathbf{b})} \otimes Z_{\Pi}^{(\mathbf{c})} \in \Gamma_{\Pi}^u(M) \otimes \Gamma_{\Pi}^{p_1-u+p_2-v}(M) \otimes \Gamma_{\Pi}^v(M).$$

We must show that the composition $\hat{\theta}_{\lambda/\mu} \circ \diamond_{\lambda/\mu}(M)(Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}) = 0$.

Using Lemma 1.4.1, (10) and (9), respectively, one may check that

$$\begin{aligned}
\Delta_{\Pi} \circ \diamond_{\lambda/\mu}(M)(Z_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}) &= \sum_{\mathbf{b}^1, \mathbf{b}^2} \text{sgn}_{Z_{\Pi}}(\mathbf{b}^1; \mathbf{b}^2) \Delta_{\Pi}(Z_{\Pi}^{(\mathbf{a})} * Z_{\Pi}^{(\mathbf{b}^1)}) \otimes \Delta_{\Pi}(Z_{\Pi}^{(\mathbf{b}^2)} * Z_{\Pi}^{(\mathbf{c})}) \\
&= \sum_{\mathbf{b}^1, \mathbf{b}^2} \sum_{\sigma} \text{sgn}_{Z_{\Pi}}(\mathbf{b}^1; \mathbf{b}^2) [\Delta_{\Pi}(Z_{\Pi}^{(\mathbf{a})}) \otimes \Delta_{\Pi}(Z_{\Pi}^{(\mathbf{b}^1)})] \cdot \sigma \otimes [\Delta_{\Pi}(Z_{\Pi}^{(\mathbf{b}^2)}) \otimes \Delta_{\Pi}(Z_{\Pi}^{(\mathbf{c})})] \cdot \sigma' \\
&= \sum_{\mathbf{b}^1, \mathbf{b}^2} \sum_{\rho, \rho'} \text{sgn}_{Z_{\Pi}}(\mathbf{b}^1; \mathbf{b}^2) Z^{\mathbf{a} \vee \mathbf{b}^1} \cdot \rho \otimes Z^{\mathbf{b}^2 \vee \mathbf{c}} \cdot \rho',
\end{aligned} \tag{21}$$

summing over all pairs of standardized sequences $\mathbf{b}^1, \mathbf{b}^2$ such that $l(\mathbf{b}^1) = p_1 - u$, $l(\mathbf{b}^2) = p_2 - v$ and $\text{st}(\mathbf{b}^1 \vee \mathbf{b}^2) = \mathbf{b}$, over all pairs of permutations $\sigma \in \mathfrak{S}_{p_1}/\mathfrak{S}_u \times \mathfrak{S}_{p_1-u}$ and $\sigma' \in \mathfrak{S}_{p_2}/\mathfrak{S}_{p_2-v} \times \mathfrak{S}_v$, and over all pairs $\rho \in \mathfrak{S}_{p_1}/\mathfrak{S}_{\mathbf{a}} \times \mathfrak{S}_{\mathbf{b}^1}$ and $\rho' \in \mathfrak{S}_{p_2}/\mathfrak{S}_{\mathbf{b}^2} \times \mathfrak{S}_{\mathbf{c}}$, respectively.

Now let $\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho')$ denote the tableau of shape λ/μ whose first row equals $(\mathbf{a} \vee \mathbf{b}^1) \cdot \rho$ and with second row $(\mathbf{b}^2 \vee \mathbf{c}) \cdot \rho'$. Let us also consider the map $\bar{\theta}_{\lambda/\mu} : \Gamma_{\Pi}^{\lambda/\mu} M \rightarrow M^{\otimes d}$ defined as the composition

$$\Gamma_{\Pi}^{\lambda/\mu} \xrightarrow{\Delta_{\Pi}} I^{\otimes d} \xrightarrow{\sigma_{\lambda/\mu}} I^{\otimes d},$$

which is the first part of the composition forming the natural transformation $\hat{\theta}_{\lambda/\mu} : \Gamma_{\Pi}^{\lambda/\mu} \rightarrow S^{\lambda'/\mu'}$. Note that $\hat{\theta}_{\lambda/\mu}(M) = m \circ \bar{\theta}_{\lambda/\mu}(M)$ where m denotes tensor product of multiplications in $S^* M$.

Since $u+v < \lambda_2 - \mu_1$, it follows that, for each $\mathbf{t} = \mathbf{t}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho')$, there exists a pair $(1, j), (2, j) \in \Delta_{\lambda/\mu}$ such that $\mathbf{t}(1, j), \mathbf{t}(2, j)$ both come from entries of \mathbf{b} . Let \check{j} denote the minimal such j . We then define an involution on the set of all $\mathbf{t}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho')$ given by $\mathbf{t} \mapsto \check{\mathbf{t}} = \check{\mathbf{t}}(\check{\mathbf{b}}^1, \check{\mathbf{b}}^2; \check{\rho}, \check{\rho}')$, where $\check{\mathbf{t}}$ is the tableau obtained from \mathbf{t} by interchanging the entries in the $(1, j)$ and $(2, j)$ positions.

For each tableau $\mathbf{t} = \mathbf{t}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho')$ as above, let us denote

$$\text{sgn}_{\pi}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho') := \text{sgn}_{Z_{\pi}}(\mathbf{b}^1, \mathbf{b}^2) \times \text{sgn}_{Z_{\pi}}(\mathbf{a} \vee \mathbf{b}^1; \rho) \times \text{sgn}_{Z_{\pi}}(\mathbf{b}^2 \vee \mathbf{c}; \rho').$$

From (5) and (21), it then remains to show that the identity

$$\text{sgn}_{\pi}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho') m(Z^{\mathbf{t}} \cdot \sigma_{\lambda/\mu}) + \text{sgn}_{\pi}(\check{\mathbf{b}}^1, \check{\mathbf{b}}^2; \check{\rho}, \check{\rho}') m(Z^{\check{\mathbf{t}}} \cdot \sigma_{\lambda/\mu}) = 0$$

holds for each $\mathbf{t} = \mathbf{t}(\mathbf{b}^1, \mathbf{b}^2; \rho, \rho')$, and this may be checked directly by computation. \square

Let $M \in \text{svec}$ with $\text{sdim}(M) = m|n$, and fix a totally ordered \mathbb{Z}_2 -homogenous basis \mathcal{Z} of M . Also write $[m+n] = [m+n]_{\mathcal{Z}}$ as above.

Theorem 3.3.10 (Standard Basis Theorem for Schur Superfunctors). *Let $\mu \subset \lambda$ be partitions such that $|\lambda/\mu| = d$. The images $\hat{\theta}_{\lambda'/\mu'}(Z_{\Pi}^{(\mathbf{t})})$ of elements indexed by costandard tableaux, $\mathbf{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$, are linearly independent and form a basis of $\hat{S}_{\lambda'/\mu'} M$. Moreover, we have an even isomorphism*

$$\hat{S}_{\lambda'/\mu'} \simeq \Gamma_{\Pi}^{\lambda/\mu} / \text{Im } \diamond_{\lambda/\mu}$$

in the category Pol_d^{I} .

Proof. It follows from the definition of $\hat{S}_{\lambda/\mu}$, Corollary 3.3.8, and Proposition 3.3.9 that there is a surjective map

$$\Gamma_{\Pi}^{\lambda/\mu} / \text{Im } \diamond_{\lambda/\mu} \twoheadrightarrow \hat{S}_{\lambda'/\mu'}$$

in the category Pol_d^{I} . Hence, the isomorphism will follow once we prove that $\hat{\theta}_{\lambda'/\mu'}(Z_{\Pi}^{(\mathbf{t})})$, with \mathbf{t} costandard, are linearly independent in $\hat{S}_{\lambda'/\mu'}(M)$.

To show linear independence, recall the surjection $m : M^{\otimes d} \twoheadrightarrow S^{\lambda'/\mu'} M$. Then the set of all

$$Z_{(\mathfrak{s})} := m(Z^{\mathfrak{s}}) \in S^d M,$$

such that $\mathfrak{s} \in \text{Tab}_{[m+n]}(\lambda'/\mu')$ is row standard, forms a basis of $S^{\lambda'/\mu'}M$. It follows from Remark 3.3.3 that \leq gives a total order on these row-standard \mathfrak{s} .

Now given any tableau $\mathfrak{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$, we let $\mathfrak{t}' \in \text{Tab}_{[m+n]}(\lambda'/\mu')$ denote the tableau of conjugate shape λ'/μ' obtained by setting $\mathfrak{t}'(i, j) = \mathfrak{t}(j, i)$ for all $(i, j) \in \Delta_{\lambda'/\mu'}$. Clearly, $\mathfrak{t} \in \text{Tab}_{[m+n]}(\lambda/\mu)$ is costandard if and only if \mathfrak{t}' is standard. Furthermore, notice that

$$Z^{\mathfrak{s}} \cdot \sigma_{\lambda'/\mu'} = \text{sgn}(\mathfrak{t}, \sigma_{\lambda'/\mu'}) Z^{\mathfrak{s}'}, \in M^{\otimes d},$$

for any $\mathfrak{s} \in \text{Tab}_{[m+n]}(\lambda/\mu)$, and the smallest element occurring in $\hat{\theta}_{\lambda'/\mu'}(Z^{(\mathfrak{t})})$ with respect to the above order, \leq , is

$$Z_{(\mathfrak{t}')} \in S^{\lambda'/\mu'}M.$$

This follows, since the action of the map $\Delta_{\Pi}(M)$ is given explicitly by

$$\Delta_{\Pi}(M) \cdot Z_{\Pi}^{(\mathfrak{t})} = \sum_{\sigma \in \mathfrak{S}_{\lambda}/\mathfrak{S}_{\mathfrak{t}}} \text{sgn}_{\pi}(\mathfrak{t}; \sigma) Z^{\mathfrak{t} \cdot \sigma}$$

where $\mathfrak{S}_{\mathfrak{t}} = \mathfrak{S}_{\mathfrak{t}^1} \times \cdots \times \mathfrak{S}_{\mathfrak{t}^q} \subset \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_q} = \mathfrak{S}_{\lambda}$. Now there is exactly one coset in $\mathfrak{S}_{\lambda}/\mathfrak{S}_{\mathfrak{t}}$ which fixes $Z_{\Pi}^{\mathfrak{t}}$. Every other representative $\sigma \in \mathfrak{S}_{\lambda}/\mathfrak{S}_{\mathfrak{t}}$ must put bigger elements of \mathfrak{t} to earlier columns. Therefore, for such σ we obtain later elements $Z_{((\mathfrak{t}, \sigma)')}$, with respect to the order \leq in $S^{\lambda'/\mu'}M$.

Moreover, one sees immediately that $Z_{(\mathfrak{t}')}$ occurs in $\hat{\theta}_{\lambda'/\mu'}(Z^{(\mathfrak{t})})$ with coefficient ± 1 . It is also obvious that the initial elements $Z_{(\mathfrak{t}')}$ are different for different costandard tableaux \mathfrak{t} . This proves that the images $\hat{\theta}_{\lambda'/\mu'}(Z^{(\mathfrak{t})})$ corresponding to costandard \mathfrak{t} are linearly independent. \square

We next wish to provide a standard basis for the evaluation of the Schur superfunctor $\hat{S}_{\lambda/\mu}^{\text{II}}$ at a supermodule $M \in \text{smod } \mathcal{C}_1$.

Let $\mathcal{U}_r(1) \in \text{smod } \mathcal{C}_1$ denote the Clifford superalgebra, \mathcal{C}_1 , considered as a supermodule over itself via right multiplication. Since \mathcal{C}_1 is a simple superalgebra, the supermodule $\mathcal{U}_r(1)$ is irreducible. Recall from [Ax, Ex. 2.6] that for every $V \in \text{smod } \mathcal{C}_1$ there is a \mathcal{C}_1 -supermodule isomorphism

$$V \simeq \mathcal{U}_r(1)^n = \mathcal{U}_r(1)^{\oplus n}$$

for some $n \geq 0$, so that $\text{sdim}(V) = n|n$.

Now fix $V \in \text{smod } \mathcal{C}_1$. Then, as a vector superspace, we have

$$\hat{S}_{\lambda/\mu}^{\text{II}}(V) = \text{Res}_{\mathcal{C}, d}(\hat{S}_{\lambda/\mu})(V) = \hat{S}_{\lambda/\mu}(F_{\mathcal{C}, d}(V)).$$

As above, we choose a totally ordered \mathbb{Z}_2 -homogeneous basis \mathcal{Z} of the superspace $F_{\mathcal{C}, d}(V)$, and let $[n + n] = [n + n]_{\mathcal{Z}}$. We again denote basis elements of $\Gamma_{\Pi}^{\lambda/\mu}(F_{\mathcal{C}}(V))$ by $Z_{\Pi}^{(\mathfrak{t})}$, for row costandard $\mathfrak{t} \in \text{Tab}_{[n+n]}(\lambda/\mu)$.

As a direct consequence of Theorem 3.3.10, we now have the following.

Corollary 3.3.11 (Standard Basis for Schur Superfunctors of Type II). *The images $\hat{\theta}_{\lambda'/\mu'}^{\text{II}}(Z_{\Pi}^{(\mathfrak{t})})$ of elements indexed by costandard tableaux, $\mathfrak{t} \in \text{Tab}_{[n+n]}(\lambda/\mu)$, form a basis of $\hat{S}_{\lambda/\mu}^{\text{II}}M$, where $\hat{\theta}_{\lambda'/\mu'}^{\text{II}} : \Gamma_{\Pi}^{\lambda'/\mu'} \rightarrow S^{\lambda/\mu}$ denotes the even morphism in Pol_d^{II} formed by composition of $\hat{\theta}_{\lambda'/\mu'}$ with the functor $F_{\mathcal{C}, d} : \Gamma_Q^d \rightarrow \Gamma_M^d$.*

4. FILTRATIONS OF BISUPERFUNCTIONS

In this section, we consider filtrations of the superspaces, $\hat{S}_{\lambda/\mu}(M \oplus N)$, for any given pair $M, N \in \text{svec}$. This leads naturally to a filtration of *Schur bisuperfunctors*.

4.1. Strict polynomial bisuperfunctors. Let $\Gamma_{\mathbf{M} \times \mathbf{M}}^d = \Gamma^d(\text{svec} \times \text{svec})$ be the category with the same objects as $\text{svec} \times \text{svec}$, i.e. pairs (M, N) of finite dimensional superspaces, and morphisms

$$\text{hom}_{\Gamma_{\mathbf{M} \times \mathbf{M}}^d}((M, N), (M', N')) = \bigoplus_{e+f=d} \text{hom}_{\Gamma_{\mathbf{M}}^e}(M, M') \otimes \text{hom}_{\Gamma_{\mathbf{M}}^f}(N, N').$$

Here, the composition of morphisms in $\Gamma_{\mathbf{M} \times \mathbf{M}}^d$ is given using the rule of signs convention (as in (1)). We then define the category $\text{biPol}_d^{\text{I}}$ of *strict polynomial bisuperfunctors* to be the category of all even \mathbb{k} -linear functors $T : \Gamma_{\mathbf{M} \times \mathbf{M}}^d \rightarrow \text{svec}$. One may also define categories $\Gamma_{\mathbf{Q} \times \mathbf{Q}}^d$ and $\text{biPol}_d^{\text{II}}$ in a similar way.

Now suppose $S \in \text{Pol}_e^{\text{I}}$, $T \in \text{Pol}_{f'}^{\text{I}}$. Then we define the *external tensor product* $S \boxtimes T : \Gamma_{\mathbf{M} \times \mathbf{M}}^{e+f} \rightarrow \text{svec}$ to be the bifunctor which acts on objects $(M, N) \in \Gamma_{\mathbf{M} \times \mathbf{M}}^{e+f}$ by

$$(S \boxtimes T)(M, N) = S(M) \otimes T(N).$$

Suppose e', f' is another pair of nonnegative integers such that $e + f = e' + f'$, and let $\varphi_1 \in \Gamma^{e'} \text{Hom}(M, N)$, $\varphi_2 \in \Gamma^{f'} \text{Hom}(M, N)$. Then, we set

$$(S \boxtimes T)(\varphi_1 \otimes \varphi_2) = \begin{cases} S(\varphi_1) \otimes T(\varphi_2) & \text{if } e' = e, f' = f \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have $S \boxtimes T \in \text{biPol}_{e+f}^{\text{I}}$.

Notice also that there is a functor $\Gamma_{\mathbf{M} \times \mathbf{M}}^d \rightarrow \Gamma_{\mathbf{M}}^d$ given by sending $(M, N) \mapsto M \oplus N$. The action on morphisms is given by the embedding

$$\text{hom}_{\Gamma_{\mathbf{M} \times \mathbf{M}}^d}((M, N), (M', N')) \hookrightarrow \text{hom}_{\Gamma_{\mathbf{M}}^d}(M \oplus N, M' \oplus N'),$$

which comes from the exponential property (7) for $\Gamma^d(\cdot)$. This induces a functor $^{\text{bi}} : \text{Pol}_d^{\text{I}} \rightarrow \text{biPol}_d^{\text{I}}$, where $T^{\text{bi}}(M, N) = T(M \oplus N)$. In a completely analogous way, we may also define a functor $^{\text{bi}} : \text{Pol}_d^{\text{II}} \rightarrow \text{biPol}_d^{\text{II}}$.

4.2. Schur bisuperfunctors. In the remainder of this section, we will study decompositions of the strict polynomial bisuperfunctors $(\hat{S}_{\lambda/\mu})^{\text{bi}}$ and $(\hat{S}_{\lambda/\mu}^{\text{II}})^{\text{bi}}$.

We denote the lexicographic order on sequences in \mathbb{Z}_+^∞ by \leq . Thus, if $\alpha, \beta \in \mathbb{Z}_+^\infty$, then $\alpha \leq \beta$ if $\alpha = \beta$ or if there exists some $1 \leq i < \infty$ such that $\alpha_1 = \beta_1, \dots, \alpha_i = \beta_i$ and $\alpha_{i+1} < \beta_{i+1}$. This total order restricts to an order on partitions which is clearly consistent with the partial order of inclusion; i.e. if $\mu \subset \lambda$, then $\mu \leq \lambda$.

If we consider $\Pi M, \Pi N$ in place of $M, N \in \text{svec}$, then recall from (7) that we have an embedding

$$\Gamma_{\Pi}^{d_1} M \otimes \Gamma_{\Pi}^{d_2} N \hookrightarrow \Gamma_{\Pi}^{d_1+d_2}(M \oplus N) \quad (22)$$

given by mapping $x \otimes y \mapsto x * y$. It follows for any partitions $\mu \subset \xi \subset \lambda$ that we have a corresponding embedding

$$\Gamma_{\Pi}^{\xi/\mu}(M) \otimes \Gamma_{\Pi}^{\lambda/\xi}(N) \hookrightarrow \Gamma_{\Pi}^{\lambda/\mu}(M \oplus N) \quad (23)$$

given by tensoring embeddings of the form (22). Notice that this further gives an Svec -enriched natural transformation: $\Gamma_{\Pi}^{\xi/\mu} \boxtimes \Gamma_{\Pi}^{\lambda/\xi} \rightarrow (\Gamma_{\Pi}^{\lambda/\mu})^{\text{bi}}$.

Definition 4.2.1. Suppose $\mu \subset \xi \subset \lambda$ are partitions such that $|\lambda/\mu| = d$. Let us define polynomial bisuperfunctors $K^\xi = K^\xi(\Gamma_{\Pi}^{\lambda/\mu})$ and $\dot{K}^\xi = \dot{K}^\xi(\Gamma_{\Pi}^{\lambda/\mu})$ as the images of the morphisms

$$\bigoplus_{\substack{\mu \subset \nu \subset \lambda, \\ \nu \leq \xi}} \Gamma_{\Pi}^{\nu/\mu} \boxtimes \Gamma_{\Pi}^{\lambda/\nu} \rightarrow (\Gamma_{\Pi}^{\lambda/\mu})^{\text{bi}}, \quad \bigoplus_{\substack{\mu \subset \nu \subset \lambda, \\ \nu < \xi}} \Gamma_{\Pi}^{\nu/\mu} \boxtimes \Gamma_{\Pi}^{\lambda/\nu} \rightarrow (\Gamma_{\Pi}^{\lambda/\mu})^{\text{bi}},$$

respectively. Then K^ξ and \dot{K}^ξ define objects in the category $\text{biPol}_d^{\text{I}}$ of strict polynomial bisuperfunctors.

Note for example that we have $\dot{K}^\xi(M, N) = \sum_{\nu < \xi} K^\nu(M, N)$ for any $M, N \in \text{svec}$.

Definition 4.2.2. With the same assumptions as in the previous definition, we define polynomial bisuperfunctors, $L_\xi = L_\xi((\hat{S}_{\lambda/\mu})^{\text{bi}})$ and $\dot{L}_\xi = \dot{L}_\xi((\hat{S}_{\lambda/\mu})^{\text{bi}})$, by setting

$$L_\xi(M, N) = \hat{\theta}_{\lambda/\mu}(K^\xi(M, N)) \quad \text{and} \quad \dot{L}_\xi(M, N) = \hat{\theta}_{\lambda/\mu}(\dot{K}^\xi(M, N)),$$

for all $M, N \in \text{svec}$. Then L_ξ and \dot{L}_ξ are also objects in $\text{biPol}_d^{\text{I}}$.

Now let $M, N \in \text{svec}$. Suppose $\text{sdim}(M) = m|n$ and $\text{sdim}(N) = m'|n'$. Let us write $r = m+n$, $s = m'+n'$. Let $\mathcal{Z} = \{Z_1, \dots, Z_r\}$ and $\mathcal{Z}' = \{Z'_1, \dots, Z'_s\}$ be \mathbb{Z}_2 -homogeneous bases of M and N , respectively, which are ordered as indicated by their indices. If we write $W_j = Z_j$ for $j = 1, \dots, r$ and $W_{r+k} = Z'_k$ for $k = 1, \dots, s$, then $\mathcal{W} = \mathcal{Z} \sqcup \mathcal{Z}' = \{W_1, \dots, W_{r+s}\}$ is an ordered homogeneous basis of $M \oplus N$. Let us introduce an indexing set for \mathcal{W} . Denote by $[r+s] = [r+s]_{\mathcal{W}}$ the set $\{1, \dots, r+s\}$, with \mathbb{Z}_2 -grading

$$[r+s]_\varepsilon = \{1 \leq i \leq r+s : |W_i| = \varepsilon\}, \quad \text{for } \varepsilon \in \mathbb{Z}_2,$$

and with the usual ordering of integers. Let $\mathbf{t} \in \text{Tab}_{[r+s]}(\lambda/\mu)$, and define the M -part of \mathbf{t} to be the sequence $\kappa(\mathbf{t})$ in $(\mathbb{Z}_{\geq 0})^\infty$ such that $\kappa(\mathbf{t})_i = \mu_i +$ the number of elements of $\{1, \dots, r\}$ in the i -th row of \mathbf{t} . If \mathbf{t} is costandard, then $\kappa(\mathbf{t})$ is a partition such that $\mu \subset \kappa(\mathbf{t}) \subset \lambda$.

Lemma 4.2.3 ([ABW], Lemma II.4.4). *Let $\mathfrak{s}, \mathbf{t} \in \text{Tab}_{[r+s]}(\lambda/\mu)$ with $\mathfrak{s} \leq \mathbf{t}$. Then $\kappa(\mathfrak{s}) \geq \kappa(\mathbf{t})$.*

We consider the subsets $[r] = \{1, \dots, r\}$ and $[s'] = \{r+1, \dots, r+s\}$ as ordered \mathbb{Z}_2 -graded subsets of $[r+s]$. Suppose ξ is a partition such that $\mu \subset \xi \subset \lambda$. If $\mathfrak{s} \in \text{Tab}_{[r]}(\xi/\mu)$ and $\mathbf{t} \in \text{Tab}_{[s']}(\lambda/\xi)$, then define the *double tableau*, $\mathfrak{s}|\mathbf{t} \in \text{Tab}_{[r+s]}(\lambda/\mu)$, by setting

$$(\mathfrak{s}|\mathbf{t})(i, j) = \begin{cases} \mathfrak{s}(i, j) & \text{if } \mu_i < j \leq \xi_i \\ \mathbf{t}(i, j) & \text{if } \xi_i < j \leq \lambda_i \end{cases}$$

for all $(i, j) \in \Delta_{\lambda/\mu}$. Clearly if \mathfrak{s}, \mathbf{t} are row (resp. column) costandard, then so is $\mathfrak{s}|\mathbf{t}$.

We will use notation similar to the previous section for the elements of $\Gamma_{\Pi}^{\lambda/\mu}(M \oplus N)$. I.e., if $\mathbf{t} \in \text{Tab}_{[r+s]}(\lambda/\mu)$, then we write

$$W_{\Pi}^{(\mathbf{t})} = (\Pi W)^{(w(\mathbf{t}))}, \quad \text{for } 1 \leq i \leq q, \quad \text{and} \quad W_{\Pi}^{(\mathbf{t})} = W_{\Pi}^{(\mathbf{t}^1)} \otimes \dots \otimes W_{\Pi}^{(\mathbf{t}^q)}$$

where $q = l(\lambda)$. If $\mathfrak{s} \in \text{Tab}_{[r]}(\xi/\mu)$ and $\mathbf{t} \in \text{Tab}_{[s']}(\lambda/\xi)$, then let us write $W_{\Pi}^{(\mathfrak{s})} * W_{\Pi}^{(\mathbf{t})} = \pm W_{\Pi}^{(\mathfrak{s}^1)} * W_{\Pi}^{(\mathbf{t}^1)} \otimes \dots \otimes W_{\Pi}^{(\mathfrak{s}^q)} * W_{\Pi}^{(\mathbf{t}^q)}$, where the coefficient ± 1 is determined as usual by the rule of signs. Then notice that the embedding (29) sends

$$W_{\Pi}^{(\mathfrak{s})} \otimes W_{\Pi}^{(\mathbf{t})} \mapsto W_{\Pi}^{(\mathfrak{s})} * W_{\Pi}^{(\mathbf{t})} = \pm W_{\Pi}^{(\mathfrak{s}|\mathbf{t})},$$

since $\text{Im}(\mathfrak{s}) \cap \text{Im}(\mathbf{t}) = \emptyset$. It follows from our definitions that $W_{\Pi}^{(\mathfrak{s}|\mathbf{t})} \in K^\xi(M, N)$. Furthermore, the set of all $W_{\Pi}^{(\mathfrak{s}|\mathbf{t})}$ corresponding to row costandard $\mathfrak{s} \in \text{Tab}_{[r]}(\nu/\mu)$, $\mathbf{t} \in \text{Tab}_{[s']}(\lambda/\nu)$, such that $\nu \leq \xi$ (resp. $\nu < \xi$), gives a basis for $K^\xi(M, N)$ (resp. $\dot{K}^\xi(M, N)$).

Proposition 4.2.4. *The canonical morphism $\Gamma_{\Pi}^{\xi/\mu} \boxtimes \Gamma_{\Pi}^{\lambda/\xi} \rightarrow K^\xi$ induces a morphism*

$$\hat{S}_{\xi/\mu} \boxtimes \hat{S}_{\lambda/\xi} \rightarrow L_\xi / \dot{L}_\xi$$

in the category $(\text{Pol}_d^{\text{I}})_{\text{ev}}$.

Proof. For each pair $M, N \in \text{svec}$, we must show that there exists a map, $\psi = \psi_\xi(M, N)$, which makes the following diagram commute:

$$\begin{array}{ccc} \Gamma_{\Pi}^{\xi/\mu} M \otimes \Gamma_{\Pi}^{\lambda/\xi} N & \xrightarrow{--*--} & K^\xi(M, N) \\ \hat{\theta}_{\xi/\mu} \boxtimes \hat{\theta}_{\lambda/\xi} \downarrow & & \downarrow \hat{\theta}_{\lambda/\mu}(\cdot \oplus \cdot) \\ \hat{S}_{\xi/\mu} M \otimes \hat{S}_{\lambda/\xi} N & \xrightarrow[\psi]{\dots\dots\dots} & L_\xi(M, N) / \dot{L}_\xi(M, N) \end{array}$$

We may use Theorem 3.3.10 to identify $\widehat{S}_{\xi/\mu}M$ with the cokernel of the map $\diamond_{\xi/\mu}(M)$, and $\widehat{S}_{\lambda/\xi}N$ with the cokernel of $\diamond_{\lambda/\xi}(N)$. In order to prove the existence of ψ it thus suffices to show that

$$\text{Im}(\diamond_{\lambda/\xi}(M)) * \Gamma_{\Pi}^{\xi/\mu}(N) + \Gamma_{\Pi}^{\lambda/\xi}(M) * \text{Im}(\diamond_{\xi/\mu}(N))$$

is contained in

$$J = \dot{K}^{\xi}(M, N) + \text{Im}(\diamond_{\lambda/\mu}(M \oplus N)).$$

We first show that $\text{Im}(\diamond_{\xi/\mu}(M)) * \Gamma_{\Pi}^{\lambda/\xi}(N)$ is contained in J . Recall that the map

$$\diamond_{\xi/\mu}(M) = \sum_{i=1}^{q-1} \sum_{u,v} 1 \otimes \cdots \otimes \diamond_i(\xi/\mu, u, v; M) \otimes \cdots \otimes 1,$$

where $q = l(\xi)$ and $u, v \geq 0$ are such that $u + v < \xi_{i+1} - \mu_i$. Let us fix such a triple i, u, v . Then choose any tableaux $\mathbf{t} \in \text{Tab}_{[s']}(\lambda/\xi)$,

$$\mathbf{r} \in \text{Tab}_{[r]}((\xi_1, \dots, \xi_{i-1})/(\mu_1, \dots, \mu_{i-1})) \quad \text{and} \quad \mathbf{s} \in \text{Tab}_{[r]}((\xi_{i+2}, \dots, \xi_q)/(\mu_{i+2}, \dots, \mu_q)).$$

We also consider a basis element $W_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}$ of $\Gamma_{\Pi}^{u, p_1 + p_2 - u - v, v} M$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in I(m|n)$ are such that $l(\mathbf{a}) = u$, $l(\mathbf{b}) = p_1 + p_2 - u - v$ and $l(\mathbf{c}) = v$, with $p_1 = \xi_i - \mu_i$ and $p_2 = \xi_{i+1} - \mu_{i+1}$. Now let us write $W = W_{\Pi}^{(\mathbf{r})} \otimes W_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})} \otimes W_{\Pi}^{(\mathbf{s})}$. Then we have

$$\diamond_{\xi/\mu}(W) = W_{\Pi}^{(\mathbf{r})} \otimes \diamond_1(\xi/\mu, u, v; M)(W_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}) \otimes W_{\Pi}^{(\mathbf{s})},$$

and it suffices to show that $\diamond_{\xi/\mu}(W) * W_{\Pi}^{(\mathbf{t})}$ belongs to J .

Next, let us consider the action of $\diamond_{\lambda/\mu}(M \oplus N)$. Let us write $\mathbf{t}_1 = (\mathbf{t}^1, \dots, \mathbf{t}^{i-1})$, $\bar{\mathbf{t}} = (\mathbf{t}^i, \mathbf{t}^{i+1})$, and $\mathbf{t}_2 = (\mathbf{t}^{i+2}, \dots, \mathbf{t}^q)$. Also consider the element

$$W' = W_{\Pi}^{(\mathbf{a})} \otimes (W_{\Pi}^{(\mathbf{b})} * W_{\Pi}^{(\mathbf{t}^i)}) \otimes (W_{\Pi}^{(\mathbf{c})} * W_{\Pi}^{(\mathbf{t}^{i+1})}).$$

Then we have

$$\begin{aligned} \diamond_{\lambda/\mu}(M \oplus N) \left((W_{\Pi}^{(\mathbf{r})} * W_{\Pi}^{(\mathbf{t}_1)}) \otimes W' \otimes (W_{\Pi}^{(\mathbf{s})} * W_{\Pi}^{(\mathbf{t}_2)}) \right) = \\ (W_{\Pi}^{(\mathbf{r})} * W_{\Pi}^{(\mathbf{t}_1)}) \otimes \diamond_i(\lambda/\mu, u, \lambda_{i+1} - \xi_{i+1} + v; M \oplus N)(W') \otimes (W_{\Pi}^{(\mathbf{s})} * W_{\Pi}^{(\mathbf{t}_2)}), \end{aligned}$$

and

$$\begin{aligned} \diamond_i(\lambda/\mu, u, \lambda_{i+1} - \xi_{i+1} + v; M \oplus N)(W') = \\ \sum_{\tilde{\mathbf{b}}^1, \tilde{\mathbf{b}}^2} \text{sgn}_{\mathcal{Z}_{\pi}}(\tilde{\mathbf{b}}^1, \tilde{\mathbf{b}}^2) (W_{\Pi}^{(\mathbf{a})} * W_{\Pi}^{(\tilde{\mathbf{b}}^1)}) \otimes (W_{\Pi}^{(\tilde{\mathbf{b}}^2)} * W_{\Pi}^{(\mathbf{c})}), \end{aligned} \quad (24)$$

where the sum is taken over all pairs of standardized $\tilde{\mathbf{b}}^1 \in I(m + m'|n + n', \lambda_i - \mu_i - u)$ and $\tilde{\mathbf{b}}^2 \in I(m + m'|n + n', \xi_{i+1} - \mu_{i+1} - v)$ such that $\text{st}(\tilde{\mathbf{b}}^1 \vee \tilde{\mathbf{b}}^2) = \mathbf{b} \vee \mathbf{t}^i$.

On the other hand, we notice that

$$\begin{aligned} \pm \diamond_{\xi/\mu}(W) * W_{\Pi}^{(\mathbf{t})} = \\ (W_{\Pi}^{(\mathbf{r})} * W_{\Pi}^{(\mathbf{t}_1)}) \otimes (\diamond_i(\xi/\mu, u, v; M)(W_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}) * W_{\Pi}^{(\mathbf{t})}) \otimes (W_{\Pi}^{(\mathbf{s})} * W_{\Pi}^{(\mathbf{t}_2)}), \end{aligned}$$

and

$$\begin{aligned} \pm \diamond_i(\xi/\mu, u, v; M)(W_{\Pi}^{(\mathbf{a}, \mathbf{b}, \mathbf{c})}) * W_{\Pi}^{(\mathbf{t})} = \\ \sum_{\mathbf{b}^1, \mathbf{b}^2} \text{sgn}_{\mathcal{Z}_{\pi}}(\mathbf{b}^1, \mathbf{b}^2) (W_{\Pi}^{(\mathbf{a})} * W_{\Pi}^{(\mathbf{b}^1)} * W_{\Pi}^{(\mathbf{t}^i)}) \otimes (W_{\Pi}^{(\mathbf{b}^2)} * W_{\Pi}^{(\mathbf{c})} * W_{\Pi}^{(\mathbf{t}^{i+1})}) \end{aligned} \quad (25)$$

where the sum is taken over all pairs of standardized $\mathbf{b}^1 \in I(m|n, \xi_i - \mu_i - u)$, $\mathbf{b}^2 \in I(m|n, \xi_{i+1} - \mu_{i+1} - v)$ such that $\text{st}(\mathbf{b}^1 \vee \mathbf{b}^2) = \mathbf{b}$.

Comparing the above sums, we see that every summand in (25), corresponding to some pair $(\mathbf{b}^1, \mathbf{b}^2)$, also appears with the same sign in (24) as the summand corresponding to the pair $(\tilde{\mathbf{b}}^1, \tilde{\mathbf{b}}^2)$, with $\tilde{\mathbf{b}}^1 = \mathbf{b}^1 \vee \mathbf{t}^i$ and $\tilde{\mathbf{b}}^2 = \mathbf{b}^2$. Furthermore, any other summand corresponding to $(\tilde{\mathbf{b}}^1, \tilde{\mathbf{b}}^2)$ which is not of the above form involves moving elements of \mathbf{t}^i from the i -th row of λ/μ to the $(i+1)$ -st row. Such a summand must belong to $\dot{K}^\xi(M, N)$. This completes the proof that $\text{Im}(\diamond_{\lambda/\xi}(M)) * \Gamma_{\Pi}^{\xi/\mu}(N)$ belongs to J . The proof that $\Gamma_{\Pi}^{\lambda/\xi}(M) * \text{Im}(\diamond_{\xi/\mu}(N)) \subset J$ is entirely symmetric. \square

Theorem 4.2.5. *For all pairs $(M, N) \in \Gamma_{M \times M}^d$, the canonical map*

$$\psi_\xi : \hat{S}_{\xi/\mu}(M) \otimes \hat{S}_{\lambda/\xi}(N) \longrightarrow L_\xi(M, N) / \dot{L}_\xi(M, N)$$

is an isomorphism. Hence, the bisuperfunctors $\{L_\xi : \mu \subset \xi \subset \lambda\}$ give a filtration of $(\hat{S}_{\lambda/\mu})^{\text{bi}}$ whose associated graded object is isomorphic to

$$\bigoplus_{\mu \subset \xi \subset \lambda} \hat{S}_{\xi/\mu} \boxtimes \hat{S}_{\lambda/\xi}.$$

Proof. First we notice that any costandard $\mathbf{t} \in \text{Tab}_{[r+s]}$ can be decomposed into a double tableau $\mathbf{t} = \mathbf{t}(1)|\mathbf{t}(2)$, where $\mathbf{t}(1)$ is of shape $\kappa(\mathbf{t})/\mu$ with elements belonging to $[r]$ and $\mathbf{t}(2)$ is the tableau of shape $\lambda/\kappa(\mathbf{t})$ obtained by restricting to the entries from the set $[s']$. It then follows from Theorem 3.3.10 that the set

$$\{\hat{\theta}_{\lambda/\mu}(W_{\Pi}^{(\mathbf{t})}) : \mathbf{t} \in \text{Tab}_{[r+s]}(\lambda/\mu) \text{ is costandard, and } \kappa(\mathbf{t}) \leq \xi \text{ (resp. } \kappa(\mathbf{t}) < \xi)\}$$

gives a basis of $L_\xi(M, N)$ (resp. $\dot{L}_\xi(M, N)$). The theorem then follows from the fact that $W_{\Pi}^{(\mathbf{t})} = \pm W_{\Pi}^{(\mathbf{t}(1))} * W_{\Pi}^{(\mathbf{t}(2))}$. \square

Notice that the functor $\text{Res}_{\mathcal{C}, d} : \text{Pol}_d^{\text{I}} \rightarrow \text{Pol}_d^{\text{II}}$ is an exact functor on the underlying even subcategories. We thus have the following consequence of Theorem 4.2.5.

Corollary 4.2.6. *For each $\mu \subset \xi \subset \lambda$, let us define the bisuperfunctor $L_\xi^{\text{II}} = \text{Res}_{\mathcal{C}, d}(L_\xi)$ in the category Pol_d^{II} . Then the set of all such L_ξ^{II} gives a filtration of $(\hat{S}_{\lambda/\mu}^{\text{II}})^{\text{bi}}$ whose associated graded object is isomorphic to*

$$\bigoplus_{\mu \subset \xi \subset \lambda} \hat{S}_{\xi/\mu}^{\text{II}} \boxtimes \hat{S}_{\lambda/\xi}^{\text{II}}.$$

5. HIGHEST WEIGHT THEORY

The main purpose of this section is to show that the Schur superfunctors $\hat{S}_\lambda^{\text{I}} = \hat{S}_\lambda$ are indecomposable objects of the category Pol_d^{I} . It will be convenient to work with the Schur superalgebra $S(m|n, d)$ for $m, n, d \in \mathbb{Z}_{\geq 0}$. We also describe formal characters of $\hat{S}_\lambda^{\text{I}}$ and $\hat{S}_\lambda^{\text{II}}$. For this, we will need to work with both Schur superalgebras $S(m|n, d)$ and $Q(n, d)$, respectively.

5.1. The Schur superalgebra $S(m|n, d)$. Suppose m, n, d are nonnegative integers, and let $M = \mathbb{k}^{m|n}$. We choose bases $\mathcal{X} = (X_1, \dots, X_m)$ of M_0 and $\mathcal{Y} = (Y_1, \dots, Y_n)$ of M_1 . Let $\mathcal{Z} = \mathcal{X} \sqcup \mathcal{Y}$ be the \mathbb{Z}_2 -graded set with $\mathcal{Z}_0 = \mathcal{X}$, $\mathcal{Z}_1 = \mathcal{Y}$ and with total order

$$X_1 < \dots < X_m < Y_1 < \dots < Y_n.$$

In this case, we write $\text{Tab}_{m|n}(\lambda/\mu) = \text{Tab}_{\mathcal{Z}}(\lambda/\mu)$ if $\mu \subset \lambda$ are any partitions. We also write $Z_i = X_i$ and $Z_{m+j} = Y_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let $E_{i,j} \in \text{End}(M)$ denote the element such that $E_{i,j}Z_k = \delta_{j,k}Z_i$ for $k = 1, \dots, m+n$. Then $\{E_{i,j} : 1 \leq i, j \leq m+n\}$ gives a basis of $\text{End}(M)$, which we order lexicographically. We define the the Schur superalgebra $S(m|n, d)$ to be the associative superalgebra

$$S(m|n, d) = \Gamma^d \text{End}(M) \cong \text{End}_{\mathbb{k}\mathfrak{S}_d}(M^{\otimes d}).$$

Denote the multiplication of $x, y \in S(m|n, d)$ by $x \circ y$.

Recall that the symmetric group \mathfrak{S}_d acts on $I(m|n, d)$ by composition. Given $\mathbf{i}, \mathbf{j} \in I(m|n, d)$, let us write $\mathbf{i} \sim \mathbf{j}$ if $\mathbf{j} = \mathbf{i} \cdot \sigma$, for some $\sigma \in \mathfrak{S}_d$. We also let \mathfrak{S}_d act on the set $I(m|n, d) \times I(m|n, d)$ via the diagonal action. We again write \sim denote the equivalence relation corresponding to the orbits of this action. E.g., if $\mathbf{i} \sim \mathbf{k}$ and $\mathbf{j} \sim \mathbf{l}$, then $(\mathbf{i}, \mathbf{j}) \sim (\mathbf{k}, \mathbf{l})$, but not vice versa. Let $\Omega(m|n, d)$ denote a set of \mathfrak{S}_n -orbit representatives in $I(m|n, d) \times I(m|n, d)$.

Following [BrKu], we say that $(\mathbf{i}, \mathbf{j}) \in I(m|n, d) \times I(m|n, d)$ is *strict* if: $(i_k, j_k) \neq (i_l, j_l)$ whenever both $k \neq l$ and $|i_k| + |j_k| = |i_l| + |j_l| = 1$. Let $I^2(m|n, d)$ denote the set of all such strict pairs. Then the diagonal action of \mathfrak{S}_d restricts to an action on $I^2(m|n, d)$. Notice that $|\mathbf{i}|_{\mathcal{Z}} + |\mathbf{j}|_{\mathcal{Z}} = |\mathbf{i}|_{\mathcal{Z}_\pi} + |\mathbf{j}|_{\mathcal{Z}_\pi}$, for any $\mathbf{i}, \mathbf{j} \in I(m|n, d)$.

We denote by $E^{(\mathbf{i}, \mathbf{j})} \in S(m|n, d) = \Gamma^d \text{End}(M)$ the element which is dual to the monomial

$$\prod_{k,l=1}^{m+n} \check{E}_{i_k, j_l} \in S^d(\text{End}(M)^*),$$

where $\{\check{E}_{i,j}\}$ denotes the dual basis of $\{E_{i,j}\}$. Then the set

$$\{E^{(\mathbf{i}, \mathbf{j})} : (\mathbf{i}, \mathbf{j}) \in \Omega(m|n, d) \cap I^2(m|n, d)\}$$

gives a basis of $S(m|n, d)$.

Given any $\varepsilon, \delta \in (\mathbb{Z}_2)^d$, we define the *charge* :

$$\text{chr}(\varepsilon, \delta) = \prod_{1 \leq s < t \leq d} (-1)^{\varepsilon_s \delta_t}.$$

Now suppose given $(\mathbf{i}, \mathbf{j}), (\mathbf{k}, \mathbf{l}) \in I^2(m|n, d)$. Then we define:

$$\text{chr}(\mathbf{i}, \mathbf{j}; \mathbf{k}) = \text{chr}(|\mathbf{i}|_{\mathcal{Z}} + |\mathbf{j}|_{\mathcal{Z}}, |\mathbf{k}|_{\mathcal{Z}}), \quad \text{chr}_\pi(\mathbf{i}, \mathbf{j}; \mathbf{k}) = \text{chr}(|\mathbf{i}|_{\mathcal{Z}} + |\mathbf{j}|_{\mathcal{Z}}, |\mathbf{k}|_{\mathcal{Z}_\pi})$$

and

$$\text{chr}(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = \text{chr}(|\mathbf{i}|_{\mathcal{Z}} + |\mathbf{j}|_{\mathcal{Z}}, |\mathbf{k}|_{\mathcal{Z}} + |\mathbf{l}|_{\mathcal{Z}}).$$

If $(\mathbf{i}, \mathbf{j}) \sim (\mathbf{k}, \mathbf{l})$, then we also define

$$\text{sgn}(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) = \text{sgn}(|\mathbf{i}|_{\mathcal{Z}} + |\mathbf{j}|_{\mathcal{Z}}, \sigma)$$

where $\sigma \in \mathfrak{S}_d$ is any permutation such that $(\mathbf{i}, \mathbf{j}) = (\mathbf{k} \cdot \sigma, \mathbf{l} \cdot \sigma)$.

We may also consider the elements $E^{\mathbf{i}, \mathbf{j}} = E^{i_1, j_1} \otimes E^{i_2, j_2} \otimes \dots \otimes E^{i_d, j_d} \in \text{End}(M)^{\otimes d} = \text{End}(M^{\otimes d})$, where the action is given by the rule of signs. Notice that we have

$$E^{\mathbf{i}, \mathbf{j}} Z^{\mathbf{k}} = \text{chr}(\mathbf{i}, \mathbf{j}; \mathbf{k}) \delta_{\mathbf{j}, \mathbf{k}} Z^{\mathbf{i}} \quad \text{and} \quad E^{\mathbf{i}, \mathbf{j}} Z_{\Pi}^{\mathbf{k}} = \text{chr}_\pi(\mathbf{i}, \mathbf{j}; \mathbf{k}) \delta_{\mathbf{j}, \mathbf{k}} Z_{\Pi}^{\mathbf{i}}. \quad (26)$$

Recall from Lemma 3.1.2 that there is an isomorphism

$$\Gamma^d \text{End}(M) = S(m|n, d) \xrightarrow{\sim} (\text{End}(M)^{\otimes d})^{\mathfrak{S}_d} = \text{End}_{\mathbb{k}\mathfrak{S}_d}(M^{\otimes d}),$$

which shows that the corresponding embedding $\Delta : S(m|n, d) \hookrightarrow \text{End}(M^{\otimes d})$ is faithful. This embedding is given explicitly by the equation

$$\Delta(E^{(\mathbf{i}, \mathbf{j})}) = \sum_{(\mathbf{s}, \mathbf{t}) \sim (\mathbf{i}, \mathbf{j})} \text{sgn}(\mathbf{i}, \mathbf{j}; \mathbf{s}, \mathbf{t}) E^{\mathbf{s}, \mathbf{t}}, \quad (27)$$

which holds for any $(\mathbf{i}, \mathbf{j}) \in I^2(m|n, d)$.

Now there is a canonical (even) isomorphism of superalgebras $\text{End}(M) \cong \text{End}(\Pi M)$. It follows that there is also a faithful embedding

$$S(m|n, d) \hookrightarrow \text{End}(\Pi M)^{\otimes d} = \text{End}((\Pi M)^{\otimes d}).$$

The superspaces $M^{\otimes d}$ and $(\Pi M)^{\otimes d}$ are thus naturally $S(m|n, d)$ -supermodules. The following lemma describes these actions (cf. [BrKu, Lem. 5.1]).

Lemma 5.1.1. (i) Suppose given $(\mathbf{i}, \mathbf{j}) \in I^2(m|n, d)$ and $\mathbf{l} \in I(m|n, d)$. Then the action of $S(m|n, d)$ on $M^{\otimes d}$ satisfies the equation

$$E^{(\mathbf{i}, \mathbf{j})} Z^{\mathbf{l}} = \sum_{(\mathbf{k}, \mathbf{l}) \sim (\mathbf{i}, \mathbf{j})} \text{sgn}(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) \text{chr}(\mathbf{i}, \mathbf{j}; \mathbf{k}) Z^{\mathbf{k}},$$

and the action on $(\Pi M)^{\otimes d}$ is similarly given by

$$E^{(\mathbf{i}, \mathbf{j})} Z_{\Pi}^{\mathbf{l}} = \sum_{(\mathbf{k}, \mathbf{l}) \sim (\mathbf{i}, \mathbf{j})} \text{sgn}(\mathbf{i}, \mathbf{j}; \mathbf{k}, \mathbf{l}) \text{chr}_{\Pi}(\mathbf{i}, \mathbf{j}; \mathbf{k}) Z_{\Pi}^{\mathbf{k}}.$$

(ii) Furthermore, the multiplication in $S(m|n, d)$ satisfies

$$E^{(\mathbf{i}, \mathbf{j})} \circ E^{(\mathbf{k}, \mathbf{l})} = \sum_{(\mathbf{s}, \mathbf{t}) \in \Omega(m|n, d)} C_{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{s}, \mathbf{t}} E^{(\mathbf{s}, \mathbf{t})},$$

with the coefficients

$$C_{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{s}, \mathbf{t}} = \sum \text{sgn}(\mathbf{i}, \mathbf{j}; \mathbf{s}, \mathbf{h}) \text{sgn}(\mathbf{k}, \mathbf{l}; \mathbf{h}, \mathbf{t}) \text{chr}(\mathbf{h}, \mathbf{t}; \mathbf{s}, \mathbf{h}),$$

the sum being over all $\mathbf{h} \in I(m|n, d)$ such that $(\mathbf{s}, \mathbf{h}) \sim (\mathbf{i}, \mathbf{j})$ and $(\mathbf{h}, \mathbf{t}) \sim (\mathbf{k}, \mathbf{l})$.

Proof. The first part of (i) follows from (26) and (27). The second equation is also not difficult to see since $|\mathbf{i}|_{\mathcal{Z}} + |\mathbf{j}|_{\mathcal{Z}} = |\mathbf{i}|_{\mathcal{Z}_{\pi}} + |\mathbf{j}|_{\mathcal{Z}_{\pi}}$.

The formula in (ii) can be verified using the first part of (i) together with the fact that the embedding $S(m|n, d) \hookrightarrow \text{End}(M^{\otimes d})$ is faithful. \square

Let us write $\Lambda(m|n) = (\mathbb{Z}_{\geq 0})^{m+n}$ and $\Lambda(m|n, d) = \{\lambda \in \Lambda(m|n) : |\lambda| = \sum \lambda_i = d\}$. We let $\Lambda^+(m|n, d)$ denote the subset consisting of all partitions $\lambda \in \Lambda(m|n, d)$. For any $\mu \in \Lambda(m|n, d)$, we define the element $E^{(\mu)} = E^{(\mathbf{i}^{\mu}, \mathbf{j}^{\mu})} \in S(m|n, d)$. The elements $E^{(\mu)}$ ($\mu \in \Lambda(m|n, d)$) are called *weight idempotents*. Given $\mathbf{i} \in I(m|n, d)$, recall the definition of $wt(\mathbf{i}) \in \Lambda(m|n, d)$ from Section 1.4. The following is a restatement of [BrKu, Lemma 5.3] in terms of our notation.

Lemma 5.1.2. Suppose $(\mathbf{i}, \mathbf{j}) \in I^2(m|n, d)$ and $\mu \in \Lambda(m|n, d)$. Then

$$E^{(\mu)} \circ E^{(\mathbf{i}, \mathbf{j})} = \begin{cases} E^{(\mathbf{i}, \mathbf{j})} & \text{if } wt(\mathbf{i}) = \mu, \\ 0 & \text{otherwise,} \end{cases} \quad E^{(\mathbf{i}, \mathbf{j})} \circ E^{(\mu)} = \begin{cases} E^{(\mathbf{i}, \mathbf{j})} & \text{if } wt(\mathbf{j}) = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\{E^{(\mu)} : \mu \in \Lambda(m|n, d)\}$ gives a set of mutually orthogonal even idempotents whose sum, $e = \sum E^{(\mu)}$, is the identity of $S(m|n, d)$.

Proof. This follows from from part (ii) of Lemma 5.1.1. \square

Given a finite dimensional $S(m|n, d)$ -supermodule, V , and $\mu \in \Lambda(m|n, d)$, we define the weight space $V_{\mu} = E^{(\mu)} V$. From the above lemma, we have a weight space decomposition

$$V = \bigoplus_{\mu \in \Lambda(m|n, d)} V_{\mu}. \quad (28)$$

We call a homogeneous vector $v \in V_{\mu}$ a *weight vector* of weight μ , and we write $wt(v) = \mu$.

5.2. Schur supermodules and Schur superfunctors. Recall from [Ax, Sec. 4] that there is a functor $\text{Pol}_d^{\mathbf{l}} \rightarrow S(m|n, d)\text{smod}$, given by sending $T \mapsto T(\mathbb{k}^{m|n})$. In particular $I^{\otimes d} \mapsto M^{\otimes d}$ and $\Pi^{\otimes d} \mapsto (\Pi M)^{\otimes d}$, where we continue to fix $M = \mathbb{k}^{m|n}$. We also have $\Gamma_{\Pi}^{\lambda} \mapsto \Gamma_{\Pi}^{\lambda}(M)$. Since there is an embedding $\Delta : \Gamma_{\Pi}^{\lambda} \hookrightarrow \Pi^{\otimes d}$, we also have an embedding

$$\Delta(M) : \Gamma_{\Pi}^{\lambda}(M) \hookrightarrow (\Pi M)^{\otimes d} \quad (29)$$

of $S(m|n, d)$ -supermodules. We wish to describe the action of $S(m|n, d)$ on $\Gamma_{\Pi}^{\lambda} M$ explicitly, by restricting the action on $(\Pi M)^{\otimes d}$ in Lemma 5.1.1 considered above.

We next define an action of the symmetric groups \mathfrak{S}_d on $\text{Tab}_{m|n}(\lambda)$. The action is given by restricting the action of \mathfrak{S}_d on $I(m|n, d)$ along the bijection, $w(\cdot) : \text{Tab}_{m|n}(\lambda) \rightarrow I(m|n, d)$,

sending a tableau to its reading word. I.e., if $\sigma \in \mathfrak{S}_d$ and $\mathbf{t} \in \text{Tab}_{m|n}(\lambda)$, then $\mathbf{t}.\sigma$ is the tableau of shape λ such that $w(\mathbf{t}.\sigma) = w(\mathbf{t}).\sigma$.

Now consider the subgroup $\mathfrak{S}_\lambda^\leftrightarrow = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_q} \subset \mathfrak{S}_d$. This subgroup acts on $\mathbf{t} \in \text{Tab}_{m|n}(\lambda)$ by permuting the entries of each row. We say that $\mathbf{s}, \mathbf{t} \in \text{Tab}_{m|n}(\lambda)$ are *row equivalent* and write $\mathbf{s} \approx_R \mathbf{t}$, if there exists $\sigma \in \mathfrak{S}_\lambda^\leftrightarrow$ such that $\mathbf{s} = \mathbf{t}.\sigma$.

We also define the subgroup permuting the elements of each column in a tableaux. For this it is convenient to consider the tableaux $\mathbf{col}_\lambda \in \text{Tab}_{m|n}(\lambda)$ given by $\mathbf{col}_\lambda(i, j) = j$ for all $(i, j) \in \Delta_\lambda$. We then let $\mathfrak{S}_\lambda^\uparrow = \{\sigma \in \mathfrak{S}_d : w(\mathbf{col}_\lambda).\sigma = w(\mathbf{col}_\lambda)\}$. Now if there exists a permutation $\sigma \in \mathfrak{S}_\lambda^\uparrow$ such that $\mathbf{s} = \mathbf{t}.\sigma$, then we say that $\mathbf{s}, \mathbf{t} \in \text{Tab}_{m|n}(\lambda)$ are *column equivalent* and write $\mathbf{s} \approx_C \mathbf{t}$.

Define a relation: $\mathbf{s} \xrightarrow{\text{RC}} \mathbf{t}$, if there exists a tableau $\mathbf{r} \in \text{Tab}_{m|n}(\lambda)$ such that $\mathbf{r} \approx_R \mathbf{s}$ and $\mathbf{r} \approx_C \mathbf{t}$.

Lemma 5.2.1. *Suppose $\mathbf{s}, \mathbf{t} \in \text{Tab}_{m|n}(\lambda)$ are both costandard. If $\mathbf{s} \xrightarrow{\text{RC}} \mathbf{t}$, then $w(\mathbf{s}) \geq w(\mathbf{t})$.*

Proof. We may assume that $\mathbf{s} \neq \mathbf{t}$. Then either $\mathbf{r} \neq \mathbf{s}$ or $\mathbf{r} \neq \mathbf{t}$. Notice that each equivalence class of \approx_C (resp. of \approx_R) contains at most one tableau which is costandard. Hence, we have both $\mathbf{r} \neq \mathbf{s}$ and $\mathbf{r} \neq \mathbf{t}$. Let i_0 be the first row of λ such that $\mathbf{r}^{i_0} \neq \mathbf{t}^{i_0}$. We must then have $\mathbf{r}(i_0, j) \geq \mathbf{t}(i_0, j)$ for $1 \leq j \leq \lambda_{i_0}$, since $\mathbf{t}(i_0, j) \leq \mathbf{t}(i, j)$, for all $i > i_0$ such that $(i, j) \in \Delta_\lambda$.

Notice that each row \mathbf{s}^i of \mathbf{s} is obtained from the i -th row of \mathbf{r} by rearranging the entries of \mathbf{r}^i into nonincreasing order. Since \mathbf{t} is costandard and $\mathbf{r}^i = \mathbf{t}^i$ for $1 \leq i < i_0$, it follows that $\mathbf{s}^i = \mathbf{t}^i$ for $1 \leq i < i_0$ as well. Now since $\mathbf{r}(i_0, j) \geq \mathbf{t}(i_0, j)$, for $1 \leq j \leq \lambda_{i_0}$, and \mathbf{s}^{i_0} is a non-increasing rearrangement of \mathbf{r}^{i_0} , we must also have $\mathbf{s}(i_0, j) \geq \mathbf{t}(i_0, j)$, for $1 \leq j \leq \lambda_{i_0}$. Finally, there must be at least one j such that $\mathbf{s}(i_0, j) > \mathbf{t}(i_0, j)$ since $\mathbf{r}^{i_0} \neq \mathbf{t}^{i_0}$. It follows that $w(\mathbf{s}) > w(\mathbf{t})$. \square

In the following, we will write $\text{chr}_\pi(\mathbf{s}, \mathbf{t}; \mathbf{t})$ instead of $\text{chr}_\pi(w(\mathbf{s}), w(\mathbf{t}); w(\mathbf{t}))$, and similar notation holds for sgn .

Lemma 5.2.2. *Suppose λ is a partition. The action of $S(m|n, d)$ on the supermodule $\Gamma_\Pi^\lambda M$ is given as follows. Suppose $\mathbf{t} \in \text{Tab}_{m|n}(\lambda)$ is row costandard. Then*

$$E^{(\mathbf{i}, \mathbf{j})} Z_\Pi^{(\mathbf{t})} = \sum_{w(\mathbf{s}) \sim \mathbf{i}} C_{\mathbf{i}, \mathbf{j}; \mathbf{s}, \mathbf{t}} Z_\Pi^{(\mathbf{s})},$$

where the sum is over row costandard $\mathbf{s} \in \text{Tab}_{m|n}(\lambda)$, with the coefficients

$$C_{\mathbf{i}, \mathbf{j}; \mathbf{s}, \mathbf{t}} = \sum \text{sgn}(\mathbf{i}, \mathbf{j}; \mathbf{s}, \mathbf{r}) \text{chr}_\pi(\mathbf{s}, \mathbf{r}; \mathbf{s}),$$

summing over all $\mathbf{r} \in \text{Tab}_{m|n}(\lambda)$ such that $\mathbf{r} \approx_R \mathbf{t}$ and $(w(\mathbf{s}), w(\mathbf{r})) \sim (\mathbf{i}, \mathbf{j})$.

Proof. Notice that the embedding (29) can be given explicitly by the formula

$$\Delta(Z_\Pi^{(\mathbf{t})}) = \sum_{\mathbf{s} \approx_R \mathbf{t}} \text{sgn}_{Z_\pi}(\mathbf{t}; \mathbf{s}) Z_\Pi^{\mathbf{s}}.$$

The lemma may then be verified by restricting the second formula in part (i) of Lemma 5.1.1 to the image of $\Gamma_\Pi^\lambda M$ under this embedding. \square

Let us order the elements of $\mathbf{i} \in I(m|n, d)$ using the lexicographic order \leq on the sequences (i_1, \dots, i_d) . I.e., we have $\mathbf{i} \leq \mathbf{j}$ if $\mathbf{i} = \mathbf{j}$ or if there exists $1 \leq k < d$ such that $i_1 = j_1, \dots, i_k = j_k$, and $i_{k+1} < j_{k+1}$.

Next we define the subsuperspace $N(m|n, d) \subset S(m|n, d)$ to be the linear span of all $E^{(\mathbf{i}, \mathbf{j})}$ such that $(\mathbf{i}, \mathbf{j}) \in I^2(m|n, d)$ and $\mathbf{i} < \mathbf{j}$. Then it is clear that $N(m|n, d)$ is a (nonunital) subalgebra. We say that an element $v \in V$, of a finite dimensional $S(m|n, d)$ -supermodule V , is $N(m|n, d)$ -invariant if $x.v = 0$ for all $x \in N(m|n, d)$. An $N(m|n, d)$ -invariant element which is also a nonzero weight vector is called a *highest weight vector* of V .

Lemma 5.2.3. *Every nonzero supermodule $V \in S(m|n, d)\text{smod}$ contains a highest weight vector.*

Proof. Suppose $v \in V_\mu$ is a weight vector, for some $\mu \in \Lambda(m|n, d)$. If $(\mathbf{i}, \mathbf{j}) \in I^2(m|n, d)$, then it is not difficult to see using Lemma 5.1.2 that $E^{(\mathbf{i}, \mathbf{j})}v$ is also a weight vector of weight $wt(\mathbf{i})$, and $E^{(\mathbf{i}, \mathbf{j})}v = 0$ unless $wt(\mathbf{j}) = \mu$. It follows that $wt(E^{(\mathbf{i}, \mathbf{j})}v) = wt(\mathbf{i}) < \mu$, whenever $\mathbf{i} < \mathbf{j}$. Hence, if we choose μ to be the earliest weight in the lexicographic order such that $V_\mu \neq 0$, then any nonzero $v \in V_\mu$ must be a highest weight vector. \square

Recall from [Ax, Sec. 4] that there is an evaluation functor $ev_{m|n}^l : \text{Pol}_d^l \rightarrow S(m|n, d)\text{smod}$, given by sending $T \mapsto T(\mathbb{k}^{m|n})$, which is an equivalence if $m, n \geq d$.

Definition 5.2.4. It follows from the preceding paragraph that the natural transformation $\hat{\theta}_\lambda : \Gamma_\Pi^\lambda \rightarrow S^\lambda$ yields by evaluation an even $S(m|n, d)$ -supermodule homomorphism

$$\hat{\theta}_\lambda(M) : \Gamma_\Pi^\lambda M \rightarrow S^\lambda M.$$

Hence, the image of this homomorphism, $\hat{S}_\lambda(M)$, is naturally an $S(m|n, d)$ -supermodule which we call the *Schur supermodule* of weight λ .

Consider the canonical tableaux \mathbf{c}_λ which is the costandard tableaux in $\text{Tab}_{m|n}(\lambda)$ defined by setting, for $(i, j) \in \Delta_\lambda$,

$$\mathbf{c}(i, j) = \begin{cases} j, & \text{if } 1 \leq j \leq m \\ m + i, & \text{if } j > m. \end{cases}$$

Then $\mathbf{c}_\lambda \leq \mathbf{t}$, for any costandard $\mathbf{t} \in \text{Tab}_{m|n}(\lambda)$.

Example 5.2.5. It follows from Lemma 5.2.3 and the Standard Basis Theorem 3.3.10 that the element $\hat{\theta}_{\lambda'}(Z_\Pi^{(\mathbf{c}_\lambda)})$ corresponding to the canonical tableau \mathbf{c}_λ is a highest weight vector of the Schur supermodule $\hat{S}_{\lambda'}(M)$.

From now on let us identify the Schur supermodule $\hat{S}_{\lambda'}M$ as the quotient $\Gamma_\Pi^\lambda M / \diamond_\lambda(M)$. For convenience, we also write $x = x + \diamond_\lambda(M)$ to denote the coset of any $x \in \Gamma_\Pi^\lambda M$. The following may be viewed as a super analogue of the result [DEP, Theorem 3.3] of De Concini, Eisenbud, and Procesi.

Theorem 5.2.6. *Let $\lambda \in \Lambda^+(m|n, d)$ be a partition such that $l(\lambda') \leq m$. Then:*

- (i) *Every nonzero $N(m|n, d)$ -submodule of $\hat{S}_{\lambda'}(M)$ contains $Z_\Pi^{(\mathbf{c}_\lambda)}$;*
- (ii) *the $N(m|n, d)$ -invariants of $\hat{S}_{\lambda'}(M)$ are spanned by $Z_\Pi^{(\mathbf{c}_\lambda)}$;*
- (iii) *$\hat{S}_{\lambda'}(M)$ is $N(m|n, d)$ -indecomposable.*

Proof. Proof of (i). Suppose $V \subset \hat{S}_{\lambda'}M$ is a nonzero $N(m|n, d)$ -submodule, and let $v \in V$. Of course, we may assume that v is not contained in the \mathbb{k} -span of $Z_\Pi^{(\mathbf{c}_\lambda)}$. Using Theorem 3.3.10, write v as a linear combination

$$v = \sum_{\mathbf{t}} a_{\mathbf{t}} Z_\Pi^{(\mathbf{t})},$$

summing over all costandard $\mathbf{t} \in \text{Tab}_{m|n}(\lambda)$, for some coefficients $a_{\mathbf{t}} \in \mathbb{k}$.

Consider any costandard \mathbf{t}_0 such that $\mathbf{t}_0 \neq \mathbf{c}_\lambda$. Then $\mathbf{c}_\lambda < \mathbf{t}_0$, so that $E^{(\mathbf{c}_\lambda, \mathbf{t}_0)} \in N(m|n, d)$. Now since $l(\lambda') \leq m$, we have $\mathbf{c}_\lambda(i, j) = j$ for all $(i, j) \in \Delta_\lambda$ (i.e. $\mathbf{c}_\lambda = \text{col}_\lambda$). Hence, if \mathbf{s} is row costandard and $w(\mathbf{s}) \sim w(\mathbf{c}_\lambda)$, then we must have $\mathbf{s} = \mathbf{c}_\lambda$. From Lemma 5.2.2, we thus have

$$E^{(w(\mathbf{c}_\lambda), w(\mathbf{t}_0))}v = \sum_{\mathbf{t}} \sum_{w(\mathbf{s}) \sim w(\mathbf{c}_\lambda)} a_{\mathbf{t}} C_{w(\mathbf{c}_\lambda), w(\mathbf{t}_0); \mathbf{s}, \mathbf{t}} Z_\Pi^{(\mathbf{s})} = \sum_{\mathbf{t}} a_{\mathbf{t}} C_{w(\mathbf{c}_\lambda), w(\mathbf{t}_0); \mathbf{c}_\lambda, \mathbf{t}} Z_\Pi^{(\mathbf{c}_\lambda)}.$$

Now if $\mathbf{r} \in \text{Tab}_{m|n}(\lambda)$, then notice that $(w(\mathbf{c}_\lambda), w(\mathbf{r})) \sim (w(\mathbf{c}_\lambda), w(\mathbf{t}_0))$ if and only if $\mathbf{r} \approx_C \mathbf{t}_0$, where we again use the fact that $l(\lambda') \leq m$. Notice first that $\mathbf{r} \approx_R \mathbf{t}_0$ and $\mathbf{r} \approx_C \mathbf{t}_0$ implies that $\mathbf{r} = \mathbf{t}_0$. Since $\text{sgn}(w(\mathbf{c}_\lambda), w(\mathbf{t}_0); \mathbf{c}_\lambda, \mathbf{t}_0) = 1$, we thus have

$$C_{w(\mathbf{c}_\lambda), w(\mathbf{t}_0); \mathbf{c}_\lambda, \mathbf{t}_0} = \text{chr}_\pi(\mathbf{c}_\lambda, \mathbf{t}_0; \mathbf{c}_\lambda).$$

On the other hand, we must have $C_{w(\mathbf{c}_\lambda), w(\mathbf{t}_0); \mathbf{c}_\lambda, \mathbf{t}} = 0$, unless $\mathbf{t} \xrightarrow{\text{RC}} \mathbf{t}_0$. It thus follows from Lemma 5.2.1 that $C_{w(\mathbf{c}_\lambda), w(\mathbf{t}_0); \mathbf{c}_\lambda, \mathbf{t}} = 0$ whenever $w(\mathbf{t}) < w(\mathbf{t}_0)$.

Let \mathbf{t}_1 be the unique costandard tableau with the property: $a_{\mathbf{t}_1} \neq 0$ and $w(\mathbf{t}_0) \geq w(\mathbf{s})$ for any costandard \mathbf{s} such that $a_{\mathbf{s}} \neq 0$. It follows from the above arguments that

$$E^{(\mathbf{c}_\lambda, \mathbf{t}_1)} v = \pm a_{\mathbf{t}_1} Z_{\Pi}^{(\mathbf{c}_\lambda)}.$$

Since $a_{\mathbf{t}_0} \neq 0$, it follows that $Z_{\Pi}^{(\mathbf{c}_\lambda)}$ is in the $N(m|n, d)$ -span of v .

Proof of (ii). The proof of (i) shows that $N(m|n, d)v \neq 0$ for any $v \in \hat{S}_\lambda M$ which is not contained in the \mathbb{k} -span of $Z_{\Pi}^{(\mathbf{c}_\lambda)}$. So (ii) follows from Example 5.2.5.

Proof of (iii). Suppose that we have a decomposition $\hat{S}_\lambda M = V \oplus W$ into $N(m|n, d)$ -submodules. From part (i) it follows that $Z_{\Pi}^{(\mathbf{c}_\lambda)}$ belongs to both V and W simultaneously, which is impossible. \square

Proposition 5.2.7. *Let λ be any partition with $d = |\lambda|$. Then the Schur superfunctor \hat{S}_λ is an indecomposable object of the category Pol_d^{I} . I.e., if $S, T \in \text{Pol}_d^{\text{I}}$ are such that $\hat{S}_\lambda = S \oplus T$, then either $S = 0$ or $S = \hat{S}_\lambda$.*

Proof. This follows from Theorem 5.2.6.(iii), since by [Ax, Thm. 4.2], the evaluation functor is an equivalence of categories, $\text{Pol}_d^{\text{I}} \xrightarrow{\sim} S(m|n, d) \text{ smod}$, for $m, n \gg 0$. \square

5.3. The Schur superalgebra $Q(n, d)$. We next discuss type II Schur superfunctors. For this, we need to consider the Schur superalgebra

$$Q(n, d) := \Gamma^d \text{End}_{\mathcal{C}_1}(\mathcal{U}_r(1)^n).$$

Let us write $V_n = \mathcal{U}(1)^n$. Recall from [Ax, Sec. 4] that there is an evaluation functor

$$\text{ev}_n^{\text{II}} : \text{Pol}_d^{\text{II}} \rightarrow Q(n, d) \text{ smod}, \quad T \mapsto T(V_n),$$

which is an equivalence if $n \geq d$. It follows that $\hat{S}_\lambda^{\text{II}}(V_n)$ has the structure of left $Q(n, d)$ -supermodule. We note that this supermodule is usually not indecomposable. Hence, $\hat{S}_\lambda^{\text{II}}$ is usually not an indecomposable object of the category Pol_d^{II} .

Now as a superspace $V_n \simeq \mathbb{k}^{n|n}$. Let us identify $\text{End}_{\mathcal{C}_1}(V_n)$ as a subset of $\text{End}(V_n)$. Then there is a corresponding superalgebra embedding $Q(n, d) \hookrightarrow S(n|n, d)$. For $1 \leq i, j \leq n$, we may consider the following elements of $\text{End}_{\mathcal{C}_1}(V_n)$:

$$E_{0; i, j} := E_{i, j} + E_{n+i, n+j} \quad \text{and} \quad E_{1; i, j} := E_{i, n+j} + E_{n+i, j}.$$

It follows from [Ax, Example 2.6] that the set $\{E_{\varepsilon; i, j} : \varepsilon \in \mathbb{Z}_2, 1 \leq i, j \leq n\}$ is a basis of $\text{End}_{\mathcal{C}_1}(V_n)$. We order this basis lexicographically: $E_{0; i, j} < E_{1; k, l}$, for all $1 \leq i, j, k, l \leq n$, and $E_{\varepsilon; i, j} \leq E_{\varepsilon; k, l}$ if $(i, j) \leq (k, l)$.

Let $I(n, d) = I(n|0, d)$. Given $\varepsilon \in (\mathbb{Z}_2)^d$ and $\mathbf{i}, \mathbf{j} \in I(n, d)$, let $E^{(\varepsilon; \mathbf{i}, \mathbf{j})}$ denote the element of $Q(n, d) = \Gamma^d \text{End}_{\mathcal{C}_1}(V_n)$ which is dual to the monomial

$$\check{E}_{\varepsilon_1; i_1, j_1} \cdots \check{E}_{\varepsilon_d; i_d, j_d} \in S^d(\text{End}_{\mathcal{C}_1}(V_n)^*),$$

where $\{\check{E}_{\varepsilon; i, j}\}$ denotes the basis of $\text{End}_{\mathcal{C}_1}(V_n)^*$ dual to the basis $\{E_{\varepsilon; i, j}\}$. If $\varepsilon \in (\mathbb{Z}_2)^d$, then we say a pair $(\mathbf{i}, \mathbf{j}) \in I(n, d) \times I(n, d)$ is ε -strict if $(i_k, j_k) \neq (i_l, j_l)$ whenever $\varepsilon_k = \varepsilon_l = 1$. Then $E^{(\varepsilon; \mathbf{i}, \mathbf{j})} \neq 0$ if and only if (\mathbf{i}, \mathbf{j}) are ε -strict.

Notice there is a unique action of the Hyperoctahedral group, $H_d = \mathbb{Z}_2 \wr \mathfrak{S}_d$, on the set $(\mathbb{Z}_2)^d \times I(n, d) \times I(n, d)$, such that: \mathfrak{S}_d acts on $I(n, d) \times I(n, d)$ via the diagonal action as usual, and $(\mathbb{Z}_2)^d \subset H_n$ acts on itself via point-wise addition. Let $\Omega(n, d)$ denote a set of H_n -orbit representatives in $(\mathbb{Z}_2)^d \times I(n, d) \times I(n, d)$. Then it is not difficult to see that the set

$$\{E^{(\varepsilon; \mathbf{i}, \mathbf{j})} : (\varepsilon; \mathbf{i}, \mathbf{j}) \in \Omega(n, d) \text{ and } (\mathbf{i}, \mathbf{j}) \text{ is } \varepsilon\text{-strict}\}$$

is a basis of $Q(n, d)$.

Next, we consider weight idempotents for $Q(n, d)$. Let us write $\Lambda(n) = \Lambda(n|0)$ and $\Lambda(n, d) = \Lambda(n|0, d)$. Denote $E^{(0;\nu)} = E^{(0;\mathbf{i}^\nu, \mathbf{i}^\nu)} \in Q(n, d)$ for any $\nu \in \Lambda(n, d)$, where $0 = (0, \dots, 0) \in (\mathbb{Z}_2)^d$. We then have the following, which is proved in [BrK1, Lem. 6.1] using different notation.

Lemma 5.3.1. *Suppose $\mathbf{i}, \mathbf{j} \in I(n, d)$, $\varepsilon \in (\mathbb{Z}_2)^d$ and $\nu \in \Lambda(n, d)$. Then*

$$E^{(0;\nu)} \circ E^{(\varepsilon;\mathbf{i}, \mathbf{j})} = \delta_{\nu, wt(\mathbf{i})} E^{(\varepsilon;\mathbf{i}, \mathbf{j})}, \quad E^{(\varepsilon;\mathbf{i}, \mathbf{j})} \circ E^{(0;\nu)} = \delta_{wt(\mathbf{j}), \nu} E^{(\varepsilon;\mathbf{i}, \mathbf{j})}.$$

In particular, $\{E^{(0;\nu)} : \nu \in \Lambda(n, d)\}$ gives a set of mutually orthogonal even idempotents whose sum is the identity of $Q(n, d)$.

Thus for any $V \in Q(n, d)\text{smod}$, we again have a decomposition

$$V = \bigoplus_{\nu \in \Lambda(n, d)} V_\nu, \tag{30}$$

where the weight space $V_\nu = E^{(0;\nu)}V$.

5.4. Formal characters. Let $X(n)$ denote the free polynomial algebra $\mathbb{Z}[x_1, \dots, x_n]$. The ring $X(n)$ has the basis of monomials: $x^\nu = x_1^{\nu_1} \dots x_n^{\nu_n}$, for $\nu \in \Lambda(n)$. Furthermore,

$$X(n) = \bigoplus_{d=0}^{\infty} X(n, d)$$

is a graded ring, where the component $X(n, d)$ is spanned by x^ν such that $\nu \in \Lambda(n, d)$.

Now let $Z(m|n)$ denote the ring $\mathbb{Z}[x_1, \dots, x_m; y_1, \dots, y_n] = X(m) \otimes Y(n)$. Then $Z(m|n)$ is also a graded ring

$$Z(m|n) = \bigoplus_{d=0}^{\infty} Z(m|n, d),$$

and $Z(m|n, d)$ has a \mathbb{Z} -basis given by the monomials

$$z^\mu = z_1^{\mu_1} \dots z_{m+n}^{\mu_{m+n}}, \quad \text{where} \quad z_1 = x_1, \dots, z_m = x_m, z_{m+1} = y_1, \dots, z_{m+n} = y_n,$$

such that $\mu \in \Lambda(m|n, d)$.

Definition 5.4.1. Suppose that $V \in S(m|n, d)\text{smod}$ and $W \in Q(n, d)\text{smod}$. Recalling the weight space decompositions (28) and (30), we define the *formal characters*

$$\text{ch}(V) = \sum_{\mu \in \Lambda(m|n, d)} \dim(V_\mu) z^\mu \quad \text{and} \quad \text{ch}(W) = \sum_{\nu \in \Lambda(n, d)} \dim(W_\nu) x^\nu,$$

which belong to $Z(m|n)$ and $X(n)$, respectively.

We now consider the weight spaces of the $S(m|n, d)$ -supermodule $\Gamma_\Pi^{\lambda'}(M)$, with $M = \mathbb{k}^{m|n}$. Let us define the weight of a tableau \mathbf{t} to be the weight of its associated word; i.e., $wt(\mathbf{t}) = wt(w(\mathbf{t}))$. Using Lemma 5.2.2 it can be checked that

$$E^{(\mu)} Z_\Pi^{(\mathbf{t})} = \delta_{wt(\mathbf{t}), \mu} Z_\Pi^{(\mathbf{t})}.$$

It follows that the weight space $(\Gamma_\Pi^{\lambda'} M)_\mu$ has basis given by the set of all $Z_\Pi^{(\mathbf{t})}$ such that \mathbf{t} is row costandard and $wt(\mathbf{t}) = \mu$. The formal character as an $S(m|n, d)$ -supermodule is then equal to

$$\text{ch}(\Gamma_\Pi^{\lambda'} M) = \sum_{\substack{\text{row costandard} \\ \mathbf{t} \in \text{Tab}_{m|n}(\lambda')}} z^{wt(\mathbf{t})} = \sum_{\substack{\text{row standard} \\ \mathbf{s} \in \text{Tab}_{m|n}(\lambda)}} z^{wt(\mathbf{s})},$$

since \mathbf{t} is costandard if and only if its conjugate, $\mathbf{s} = \mathbf{t}'$, is standard. (Recall that $\mathbf{t}'(i, j) = \mathbf{t}(j, i)$ for all $(i, j) \in \Delta_{\lambda'}$.) It then follows from Theorem 3.3.10 that

$$\text{ch}(\hat{S}_\lambda(\mathbb{k}^{m|n})) = \sum_{\substack{\text{costandard} \\ \mathbf{t} \in \text{Tab}_{m|n}(\lambda')}} z^{wt(\mathbf{t})} = \sum_{\substack{\text{standard} \\ \mathbf{s} \in \text{Tab}_{m|n}(\lambda)}} z^{wt(\mathbf{s})}. \tag{31}$$

Suppose given partitions $\mu \subset \lambda$. Then recall from [Mac, ch. I] the definition of the *skew Schur function*, $s_{\lambda/\mu}(x_1, \dots, x_n)$, which belongs to the ring $\Lambda_n(x) = X(n)^{\mathfrak{S}_n}$ of symmetric functions in n variables. One may check using [Mac, I.5.12] that we have

$$s_{\lambda/\mu}(x_1, \dots, x_m) = \sum_{\substack{\text{standard} \\ \mathbf{t} \in \text{Tab}_{m|0}(\lambda/\mu)}} z^{wt(\mathbf{t})}, \quad s_{\lambda'/\mu'}(y_1, \dots, y_n) = \sum_{\substack{\text{standard} \\ \mathbf{s} \in \text{Tab}_{0|n}(\lambda'/\mu')}} z^{wt(\mathbf{s})}. \quad (32)$$

The *hook Schur function* is defined in [BR, 6.3] to be the sum

$$hs_{\lambda}(x_1, \dots, x_m; y_1, \dots, y_n) = \sum_{\mu \subset \lambda} s_{\mu}(x_1, \dots, x_m) s_{\lambda'/\mu'}(y_1, \dots, y_n),$$

which may be considered as an element of the ring $\Lambda_m(x) \otimes_{\mathbb{Z}} \Lambda_n(y) \subset Z(m|n)$.

Proposition 5.4.2. *The character of the Schur supermodule $\hat{S}_{\lambda}(\mathbb{k}^{m|n})$ is equal to the hook Schur function $hs_{\lambda}(x_1, \dots, x_m; y_1, \dots, y_n)$.*

Proof. Any standard tableaux $\mathbf{t} \in \text{Tab}_{m|n}(\lambda)$ may be decomposed as a double tableau $\mathbf{t}(1)|\mathbf{t}(2)$ for standard $\mathbf{t}(1) \in \text{Tab}_{m|0}(\mu)$ and $\mathbf{t}(2) \in \text{Tab}_{0|n}(\lambda/\mu)$ for some $\mu \subset \lambda$. Hence, the result follows from (31) and (32). \square

Recall from [Mac, Ch. III], [WW] the *Hall-Littlewood symmetric function*,

$$S_{\lambda}(x_1, \dots, x_n) = hs_{\lambda}(x_1, \dots, x_n; x_1, \dots, x_n) = \sum_{\mu \subset \lambda} s_{\mu}(x_1, \dots, x_m) s_{\lambda'/\mu'}(x_1, \dots, x_n),$$

defined for $\lambda \in \Lambda(n)$. We then have the following.

Corollary 5.4.3. *The character of the $Q(n, d)$ -supermodule $\hat{S}_{\lambda}^{\text{II}}(V_n)$ is equal to the Hall-Littlewood symmetric function $S_{\lambda}(x_1, \dots, x_n)$.*

Proof. Given any $\mu \in \Lambda(m|n)$, let us write $\mu^+ = (\mu_1, \dots, \mu_m) \in \Lambda(m)$ and $\mu^- = (\mu_{m+1}, \dots, \mu_{m+n}) \in \Lambda(n)$. Then note that the embedding $Q(n, d) \hookrightarrow S(n|n, d)$ mentioned above sends

$$E^{(0;\nu)} \mapsto \sum_{\mu \in \Lambda(n|n, d): \mu^+ + \mu^- = \nu} E^{(\mu)}.$$

Now any $V \in S(n|n, d)\text{smod}$ may be considered as a $Q(n, d)$ -supermodule by restriction. In particular, for any $\nu \in \Lambda(n, d)$ we have

$$\dim(\text{Res}_{Q(n, d)}^{S(n|n, d)}(V)_{\nu}) = \sum_{\mu \in \Lambda(n|n, d): \mu^+ + \mu^- = \nu} \dim(V_{\mu}).$$

Then since $x^{\nu} = x^{\mu^+} x^{\mu^-}$ whenever $\mu^+ + \mu^- = \nu$, it follows that $\text{ch}(\text{Res}_{Q(n, d)}^{S(n|n, d)}(V)) = \text{ch}(V)|_{x=y}$. The result now follows from Proposition 5.4.2 and the definitions since

$$\hat{S}_{\lambda}^{\text{II}}(V_n) = \text{Res}_{Q(n, d)}^{S(n|n, d)} \hat{S}_{\lambda}(\mathbb{k}^{n|n})$$

as a $Q(n, d)$ -supermodule. \square

Let us briefly describe formal characters of polynomial superfunctors. Let $m' \geq m$, $n' \geq n$ be nonnegative integers. Then there are surjective ring homomorphisms

$$\rho_n^{n'} : X(n') \rightarrow X(n), \quad \text{resp.} \quad \rho_{m|n}^{m'|n'} : Z(m|n') \rightarrow Z(m|n),$$

which send the variables $x_{n+1}, \dots, x_{n'}$, resp. $x_{m+1}, \dots, x_{m'}, y_{n+1}, \dots, y_{n'}$, to zero and which leave other variables fixed. These maps restrict to give surjective maps of abelian groups:

$$\rho_n^{n'}(d) : X(n', d) \rightarrow X(n, d), \quad \rho_{m|n}^{m'|n'}(d) : Z(m'|n', d) \rightarrow Z(m|n, d).$$

We may define inverse limits,

$$X(\infty, d) = \varprojlim X(n, d) \quad \text{and} \quad Z(\infty|\infty, d) = \varprojlim Z(m|n, d),$$

with respect to these maps. We then have corresponding graded rings,

$$X(\infty) = \bigoplus_{d=0}^{\infty} X(\infty, d), \quad \text{and} \quad Z(\infty|\infty) = \bigoplus_{d=0}^{\infty} Z(\infty|\infty, d),$$

with multiplication induced from the inverse limits.

Notice that $X(\infty)$ and $Z(\infty|\infty)$ may both be considered as subrings of the ring $\mathbb{Z}[[x; y]] = \mathbb{Z}[[x_1, x_2, \dots; y_1, y_2, \dots]]$ of formal power series in infinitely many variables. Explicitly, we may identify an element

$$(f_n)_{n \geq 0} \in X(\infty, d), \quad \text{resp.} \quad (g_{m|n})_{m, n \geq 0} \in Z(\infty|\infty, d),$$

with the unique power series $f(x)$, resp. $g(x; y)$, such that $f_n(x_1, \dots, x_n) = f(x_1, \dots, x_n, 0, \dots)$ and $g_{m|n}(x_1, \dots, x_n; y_1, \dots, y_n) = g(x_1, \dots, x_m, 0, \dots; y_1, \dots, y_n, 0, \dots)$ for all $m, n \geq 0$. In this case, we write

$$f = \varprojlim f_n, \quad g = \varprojlim g_{m|n}.$$

For example, we may consider inverse limits of the Hall-Littlewood and hook Schur functions:

$$S_\lambda(x) = \varprojlim S_\lambda(x_1, \dots, x_n), \quad hs_\lambda(x; y) = \varprojlim hs_\lambda(x_1, \dots, x_m; y_1, \dots, y_n).$$

Note that we have: $S_\lambda(x; y) = hs_\lambda(x; x)$, and $hs_\lambda(x; y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda/\mu'}(y)$, where $s_\lambda(x)$ is the Schur function $s_\lambda(x) = \varprojlim s_\lambda(x_1, \dots, x_n) \in X(\infty)$.

We remark that it is possible to show that

$$\rho_{m|n}^{m'|n'}(\text{ch}(T(\mathbb{k}^{m'|n'}))) = \text{ch}(T(\mathbb{k}^{m|n}))), \quad \text{and} \quad \rho_n^{n'}(\text{ch}(T(V_{n'}))) = \text{ch}(T(V_n))$$

for all $m' \geq m, n' \geq n$. Hence, one may make the following definition.

Definition 5.4.4. The *formal character* of a polynomial superfunctor $T \in \text{Pol}_d^I$, resp. $T \in \text{Pol}_d^{II}$, is defined to be the element

$$\text{ch}^I(T) = \varprojlim \text{ch}(T(\mathbb{k}^{m|n})), \quad \text{resp.} \quad \text{ch}^{II}(T) = \varprojlim \text{ch}(T(V_n)),$$

of $X(\infty, d)$, resp. $Z(\infty|\infty, d)$.

The Schur superfunctors then have the following formal characters:

$$\text{ch}^I(\hat{S}_\lambda^I) = hs_\lambda(x; y) \quad \text{and} \quad \text{ch}^{II}(\hat{S}_\lambda^{II}) = S_\lambda(x), \quad (33)$$

which follows from the definitions, Proposition 5.4.2 and Corollary 5.4.3.

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